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AND

DISCUSSIONS

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AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

PAPERS

ANALYSIS OF VIERENDEEL TRUSSES

By Dana Young,[1] Assoc. M. Am. Soc. C. E.

Synopsis

A Vierendeel or quadrangular truss is a hyperstatic frame composed of a series of rectangular or trapezoidal panels without diagonal members. The end connections of all members are rigid and designed to take moment.

Although it is possible to analyze such trusses by any of the so-called "exact" methods, the labor involved is prohibitive in all except the simplest cases. The purpose of this paper is to present a practical and accurate method of determining the stresses in the Vierendeel type of trusses. This method is based upon the theory of virtual work and involves a re-arrangement of the standard equations to produce a workable solution.

The interesting questions of design, economics, and field of usefulness of such trusses are not considered, the scope of this paper being limited to stress analysis.

Historical

Vierendeel trusses have been used for bridges in Europe, chiefly in Belgium, since 1896. The first bridge of this type was built at Tervueren, Belgium, by Professor Arthur Vierendeel of the University of Louvain. Professor Virendeel is the originator of this form of truss and has been its main sponsor. To date (1936), approximately one hundred bridges of this type have been built. The longest has a span of 274 ft and carries a double-track railway. Both steel and reinforced concrete have been used and, since 1932, several arc-welded steel trusses of this type have been built, including the trusses for a swing bridge.[2]

In addition, various other structures using this system have been constructed. Eight wireless towers, each 960 ft high and only 13 ft square in plan, with bracing of the Vierendeel type, have been built in Belgium. Towers for elevated water tanks have been similarly braced.

Note.—Discussion on this paper will be closed in **October, 1936,** *Proceedings.*

[1] Asst. Prof. of Eng., Connecticut State Coll., Storrs, Conn.

[2] "Vierendeel Truss Bridges Popular in Belgium", by L. C. Rucquoi, M. Am. Soc. C. E., *Engineering News-Record,* July 25, 1935, p. 116.

In the United States, Vierendeel trusses have not been used for bridges, but a number of other structures have been built with this system of bracing. The towers of the old Kinzua Viaduct of the Erie Railroad Company were of that form.[3] Various reinforced concrete viaduct bents have been built utilizing this construction. The towers of the Waldo-Hancock Suspension Bridge in Maine are braced transversely as Vierendeel trusses.[4] A modified form of Vierendeel truss was used for the foundation of the Telephone Building[5], at Albany, N. Y.

Introduction to Stress Analysis

In the general type of Vierendeel truss, with n panels and $(n + 1)$ web members, there are $3\ (n + 1)$ unknowns. It is necessary to obtain an equal number of equations to solve the problem. The equations of statics furnish three relations and the remainder are found from the elastic properties of the truss.

Any of the standard methods of elastic analysis may be used. In books on structural theory methods of solution will be found for a few limited cases, using the principles of least work[6], virtual work[7], slope deflection[7], and moment distribution.[8] However, in all except the simplest cases, the application of these methods is laborious.

The aim of the writer is to describe a method of analysis which is less laborious to apply. It is based upon the principles of virtual work and consists in re-arranging the fundamental equations, and the order of their solution. For trusses with chords that are symmetrical about a longitudinal axis (such as parallel chord trusses and viaduct bents), the solution is exact; that is, it involves no assumptions except those that are inherent in the elastic theory. For other trusses it is necessary to make use of certain simplifying assumptions. In most cases the relative error thus introduced is negligible.

The first part of the analysis follows, in general outline, the method developed by Professor Vierendeel.[9] However, in order to obtain greater accuracy, this analysis does not involve all the simplifying assumptions that Professor Vierendeel uses. As a result, the detailed treatment and final formulas are different.

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[3] "The Kinzua Viaduct of the Erie Railroad Company", by C. R. Grimm, M. Am. Soc. C. E., *Transactions*, Am. Soc. C. E., Vol. XLVI (1901), pp. 21–77.

[4] "Building First Long Span Bridge in Maine", by D. B. Steinman, M. Am. Soc. C. E., and C. H. Gronquist, Assoc. M. Am. Soc. C. E., *Engineering News-Record*, March 17, 1932, p. 386.

[5] "Foundation for Albany Telephone Building", by G. W. Glick, *Engineering News-Record*, November 27, 1930, p. 836.

[6] "Elastic Energy Theory", by J. A. Van den Broek, M. Am. Soc. C. E., John Wiley & Sons, New York, pp. 81–87.

[7] "Modern Framed Structures", Pt. II, by J. B. Johnson, C. W. Bryan, and F. E. Turneaure, John Wiley & Sons, New York, 1926.

[8] "Continuous Frames of Reinforced Concrete", by Hardy Cross and N. D. Morgan, Members, Am. Soc. C. E., John Wiley & Sons, New York, pp. 236–238; also, see, "A Rapid and Concise Method of Analyzing Rigid Viaduct Bents", by L. C. Maugh, Assoc. M. Am. Soc. C. E., *Engineering News-Record*, March 14, 1935, p. 379.

[9] "Cours de Stabilité des Construction", Tome IV, by Arthur Vierendeel, Louvain, 1920, and succeeding editions. This text contains Professor Vierendeel's approximate method of solution and a bibliography on the subject. The references in the bibliography are mainly to European publications, few of which are available to engineers in the United States.

Notation.—The letter symbols in this paper are introduced in the text as they occur and are summarized for reference in Appendix I. An effort has been made to conform essentially with "Symbols for Mechanics, Structural Engineering and Testing Materials"[10], compiled by a committee of the American Standards Association, with Society representation, and approved by the Association in 1932.

Analysis of Trusses with Symmetrical Chords

By a truss with symmetrical chords is meant one in which the chords are symmetrical about a longitudinal axis, with the corresponding chord members having equal moments of inertia. This class includes parallel-chord trusses and viaduct bents with inclined posts.

For the present, deformations due to axial stress will be neglected. Subsequently, the analysis will be extended to include this effect. Fig. 1 represents

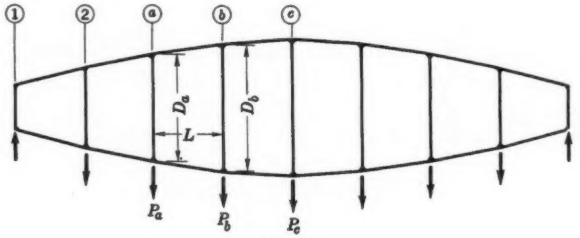

FIG. 1.

a general form of such a truss. From the symmetry of the truss, it is seen that the bending moments and shears in the upper and lower chords will be equal. It also follows that the point of contraflexure of each vertical member will be at its mid-height.

Consider a typical panel, *ab*, removed from the truss and cut into two parts by a horizontal plane through the centers of the vertical members. Let Δ_x = horizontal deflection; L_i = the length of an inclined chord member; H_w = transverse shear in a web member; H_{ab} = horizontal component of axial

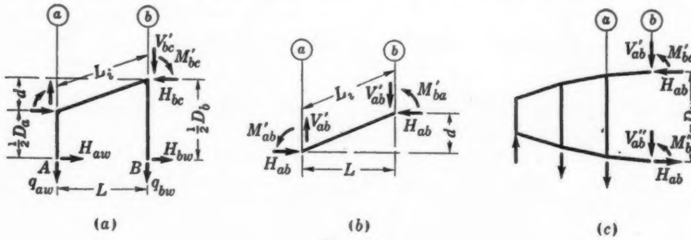

FIG. 2.

stress in Chord Member *a-b*; D_a = depth of truss at Panel Point *a*, etc.; V'_{ab} = vertical shear in the upper chord between Panel Points *a* and *b*, etc.; M''_{bc} = bending moment at Point *b* in the lower chord member, *b-c*

[10] A. S. A.—Z10*a*—1932.

(that is, the member extending between Panel Points b and c), etc. The forces acting on the upper part are shown in Fig. 2(a). For this section the horizontal displacement of Point B with respect to Point A is:

$$E\,\Delta_x = \int \frac{M\,y}{I}\,ds = \frac{1}{I_b}\int_0^{0.5 D_b} H_{bw}\,y^2\,dy + \frac{1}{I_C}\int_0^{L_i}\left[H_{bw}\left(\frac{D_b}{2} - y\right) - M'_{bc}\right.$$

$$\left. - (V'_{bc} + q_{bw})\,x + H_{bc}\,y\right]\left(\frac{D_b}{2} - y\right) dL_i + \frac{1}{I_a}\int_0^{0.5 D_a} - H_{aw}\,y^2\,dy \;\ldots(1)$$

The values of the first and third integrals of Equation (1) are:

$$\int_0^{0.5 D_b} H_{bw}\,y^2\,dy = \frac{H_{bw}\,y^3}{3}\Bigg]_0^{0.5 D_b} = \frac{H_{bw}\,D_b^3}{24} \;\ldots\ldots\ldots\ldots(2a)$$

and,

$$\int_0^{0.5 D_a} - H_{aw}\,y^2\,dy = \frac{H_{aw}\,y^3}{3}\Bigg]_0^{0.5 D_a} = -\frac{H_{aw}\,D_a^3}{24} \;\ldots\ldots\ldots(2b)$$

The forces acting on the upper chord are shown in Fig. 2(b). Comparing these forces with the forces in Fig. 2(a), it is obvious that:

$$H_{ab} = H_{bc} - H_{bw} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3a)$$

$$V'_{ab} = V'_{bc} + q_{bw} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3b)$$

and,

$$M'_{ba} = \frac{H_{bw}\,D_b}{2} - M'_{bc} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3c)$$

Substitute Equations (3) in the second integral of Equation (1) and perform the integration. In the resulting expression substitute $d = \frac{1}{2}\,(D_b - D_a)$ and $H_{ab} = \sum_1^a H_w$, and this integral finally becomes:

$$\int_0^{L_i} (M'_{ba} - V'_{ab}\,x + H_{ab}\,y)\left(\frac{D_b}{2} - y\right) dL_i = \frac{M'_{ba}\,L_i}{4}\,(D_b + D_a)$$

$$- V'_{ab}\,\frac{L_i L}{12}\,(D_b + 2\,D_a) + \frac{L_1}{24}\,(D_b^2 + D_b\,D_a - 2\,D_a^2)\sum_1^a H_w \ldots(4)$$

The integral values in Equations (2a), (2b), and (4) substituted in Equation (1), give Equation (5) directly, as follows:

$$E\,\Delta_x = \frac{D_b^3\,H_{bw}}{24\,I_b} - \frac{D_a^3\,H_{aw}}{24\,I_a} + \frac{L_i\,M'_{ba}}{4\,I_C}\,(D_a + D_b) - \frac{L_i\,L\,V'_{ab}}{12\,I_C}\,(D_b + 2\,D_a)$$

$$+ \frac{L_i}{24\,I_C}\,(D_b^2 + D_b\,D_a - 2\,D_a^2)\sum_1^a H_w \;\ldots\ldots\ldots\ldots(5)$$

From symmetry, it is seen that $\Delta_x = 0$. Now, consider that part of the entire truss which is to the left of Panel Point b, as shown in Fig. 2(c).

Taking moments about Panel Point b:

$$M_b - M'_{ba} - M''_{ba} - H_{ab} D_b = 0 \quad \text{.......... (6)}$$

and, since $M'_{ba} = M''_{ba}$ and $H_{ab} = \sum_1^a H_w$, Equation (6) becomes:

$$M'_{ba} = \frac{1}{2} M_b - \frac{1}{2} D_b \sum_1^a H_w \quad \text{.......... (7)}$$

It is obvious that:

$$V'_{ab} + V''_{ab} = 2\ V'_{ab} = V_{ab} \quad \text{.......... (8)}$$

Substituting these values for M'_{ba} and V'_{ab} in Equation (5) and collecting terms:

$$E\ \Delta_x = 0 = \frac{D^3_b H_{bw}}{24\, I_b} - \frac{D^3_a H_{aw}}{24\, I_a} + \frac{L_t M_b}{8\, I_C}(D_a + D_b) - \frac{L_t L V_{ab}}{24\ I_C}(D_b + 2\ D_a)$$

$$- \frac{L_t}{12\, I_C}(D^2_b + D_b D_a + D^2_a) \sum_1^a H_w \quad \text{.......... (9)}$$

Solving for H_{bw}:

$$H_{bw} = \frac{D^3_a I_b H_{aw}}{D^3_b I_a} + \frac{2\, L_t I_b}{D^3_b I_C}(D^2_b + D_b D_a + D^2_a) \sum_1^a H_w - \frac{3\, L_t I_b M_b}{D^3_b I_C}(D_a + D_b)$$

$$+ \frac{L_t L I_b V_{ab}}{D^3_b I_C}(D_b + 2\ D_a) \quad \text{.......... (10)}$$

For parallel chords and constant I, Equation (10) reduces to:

$$H_{bw} = H_{aw} + 6\, \frac{L}{D} \sum_1^a H_w - \frac{6\ L}{D^2} M_b + \frac{3\ L^2}{D^2} V_{ab} \quad \text{.......... (11)}$$

Equation (10) is the general equation for trusses with symmetrical chords. All the coefficients involved are constants which depend only on the dimensions of the truss. Hence, Equation (10) is really of the simple form:

$$H_{bw} = C_1 H_{aw} + C_2 \sum_1^a H_w + C_3 M_b + C_4 V_{ab} \quad \text{.......... (12)}$$

and since M_b and V_{ab} are constants which depend on the external loads, Equation (12) may be expressed in the simpler form:

$$H_{bw} = C_1 H_{aw} + C_2 \sum_1^a H_w + C_5 \quad \text{.......... (13)}$$

An expression of the form of Equation (13) may be written for each panel in the truss. Then, by successive substitutions, each value of H_w may be expressed in terms of the shear, H_{1w}, for the first vertical. Since the sum

of all the H_w-values is equal to zero, it is a simple matter to solve for the value of H_{1w} and thence for the other values of H_w. The following two examples illustrate this method of solution.

Example 1.—Parallel Chord Truss.—Consider a four-panel, parallel chord truss with a height of 10 ft and a span of 40 ft, carrying 1 000 lb at each panel point, as shown in Fig. 3. The moment of inertia of all members is assumed

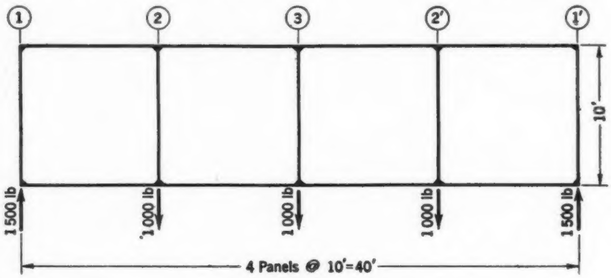

FIG. 3.

to be the same. Substituting numerical values for this particular case in Equation (11):

$$H_{bw} = H_{aw} + 6 \sum_{1}^{a} H_w - 0.6\, M_b + 3\, V_{ab} \qquad (14)$$

It should be noted that Equation (14) remains the same for any loading, since the coefficients depend only on the truss dimensions. The values for M_b and V_{ab} for this particular loading are, respectively: $M_2 = 15\,000$ ft-lb; $M_3 = 20\,000$ ft-lb; $V_{12} = 1\,500$ lb; and $V_{23} = 500$ lb. Substituting these values in Equation (14):

$$H_{1w} = H_{1w} \qquad (15a)$$

$$H_{2w} = H_{1w} + 6\, H_{1w} - 9\,000 + 4\,500 = 7\, H_{1w} - 4\,500 \qquad (15b)$$

and,

$$H_{3w} = H_{2w} + 6\,(H_{1w} + H_{2w}) - 12\,000 + 1\,500 \qquad (15c)$$

Substituting Equation (15*b*) in Equation (15*c*):

$$H_{3w} = 55\, H_{1w} - 42\,000 \qquad (16)$$

From the symmetry of the loading in this particular case, it is obvious that $H_{3w} = 0$; hence:

$$H_{3w} = 55\, H_{1w} - 42\,000 = 0 \qquad (17)$$

Solving for H_{1w}: $H_{1w} = \dfrac{42\,000}{55} = 763.6$ lb, and, thence, $H_{2w} = 7\ (763.6) - 4\,500 = 845$ lb.

Example 2.—Viaduct Bent.—As a second example, a bent of the Kinzua Viaduct will be studied. This particular structure has been analyzed by Mr. C. R. Grimm[8], using the method of least work, and by Messrs Johnson, Bryan, and Turneaure[11], using the general method of virtual work. Their solutions afford a check on the method presented herein.

Fig. 4 shows the dimensions of the bent and the lateral loads to be considered. The bases of the posts are assumed to be fixed. This is equivalent to assuming that the shear, H_{6w}, in the bottom strut is zero. Equation (10) applies in this problem. Evaluating the coefficients in this equation by substituting the numerical values of the truss dimensions, the following series of equations is obtained from Equation (10):

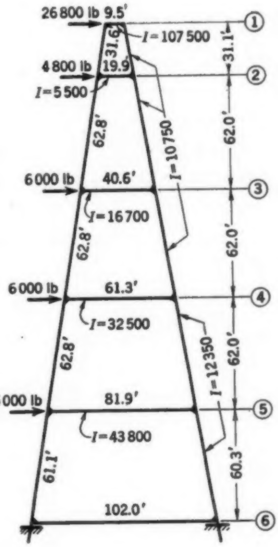

FIG. 4.

$$H_{1w} = H_{1w} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (18a)$$

$$H_{2w} = 0.00557\ H_{1w} + 2.77\ H_{1w} - 0.1810\ M_2 + 2.50\ V_{12} \ldots\ldots (18b)$$

$$H_{3w} = 0.358\ H_{2w} + 8.30\ (H_{1w} + H_{2w}) - 0.264\ M_3 + 7.26\ V_{23} \ldots\ldots (18c)$$

$$H_{4w} = 0.565\ H_{3w} + 13.02\ (H_{1w} + H_{2w} + H_{3w}) - 0.252\ M_4 + 7.29\ V_{34} \ldots (18d)$$

$$H_{5w} = 0.565\ H_{4w} + 12.52\ (H_{1w} + H_{2w} + H_{3w} + H_{4w}) - 0.1745\ M_5 + 5.14\ V_{45} \ldots\ldots\ldots\ldots (18e)$$

and,

$$H_{6w} = 0 = 1.043\ H_{5w} + 21.0\ (H_{1w} + H_{2w} + H_{3w} + H_{4w} + H_{5w}) - 0.227\ M_6 + 6.60\ V_{56} \ldots\ldots\ldots\ldots (18f)$$

Equations (18) are true for any loading. For the loading in Example 2, M_b and V_{ab} have the following values: M_2 = 835 000 ft-lb; M_3 = 2 798 000; M_4 = 5 128 000; M_5 = 7 830 000; M_6 = 10 825 000; V_{12} = 26 800 lb; V_{23} = 31 600; V_{34} = 37 600; V_{45} = 43 600; and V_{56} = 49 600. Substituting these numerical values in Equations (18):

$$H_{1w} = H_{1w} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (19a)$$

$$H_{2w} = 0.00557\ H_{1w} + 2.77\ H_{1w} - 84\,100 \ldots\ldots\ldots\ldots\ldots\ldots (19b)$$

$$H_{3w} = 0.358\ H_{2w} + 8.30\ (H_{1w} + H_{2w}) - 509\,000 \ldots\ldots\ldots\ldots (19c)$$

[11] "Modern Framed Structures", Pt. II, by J. B. Johnson, C. W. Bryan, and F. E. Turneaure, John Wiley & Sons, New York, 1926, pp. 353–354.

$$H_{4w} = 0.565\ H_{3w} + 13.02\,(H_{1w} + H_{2w} + H_{3w}) - 1\,018\,000 \quad \text{(19}d\text{)}$$

$$H_{5w} = 0.565\ H_{4w} + 12.52\ (H_{1w} + H_{2w} + H_{3w} + H_{4w}) - 1\,142\,000 \quad \text{(19}e\text{)}$$

and,

$$0 = 0.0497\ H_{5w} + 1.00\ (H_{1w} + H_{2w} + H_{3w} + H_{4w} + H_{5w}) - 101\,300 \quad \text{(19}f\text{)}$$

To this point, slide-rule calculations have been used; but now the computations must be carried to more places. This is not false accuracy. It is an arithmetical procedure which is necessary because of the subtractions that occur in solving the equations.

By successive substitutions, each H_w may be expressed in terms of H_{1w}; thus:

$$H_{1w} = H_{1w} \quad \text{(20}a\text{)}$$

$$H_{2w} = 2.776\ H_{1w} - 84\,100 \quad \text{(20}b\text{)}$$

$$H_{3w} = 32.33461\ H_{1w} - 1\,237\,138 \quad \text{(20}c\text{)}$$

$$H_{4w} = 488.4292\ H_{1w} - 18\,919\,500 \quad \text{(20}d\text{)}$$

$$H_{5w} = 6\,843.200\ H_{1w} - 265\,245\,500 \quad \text{(20}e\text{)}$$

and,

$$0 = 7\,707.8473\ H_{1w} - 298\,770\,300 \quad \text{(20}f\text{)}$$

Solving Equation (20*f*) to determine H_{1w}:

$H_{1w} = \dfrac{298\,770\,300}{7707.847} = 38\,761.83$. Substituting this value of H_{1w} in Equations (20*a*) to (20*e*) supplies the remaining values of H_w, thus: $H_{2w} = 23\,500$ lb; $H_{3w} = 16\,200$; $H_{4w} = 12\,900$; and, $H_{5w} = 9\,400$.

The corresponding values, as found by Johnson, Bryan, and Turneaure, are: $H_{1w} = 38\,900$; $H_{2w} = 23\,400$; $H_{3w} = 16\,300$; $H_{4w} = 12\,400$; and $H_{5w} = 9\,900$. It is seen that these values are in close agreement with those computed by the writer. The differences are due to the slide-rule calculations of the coefficients in this Example.

Analysis of Trusses with Inclined Upper Chords

Trusses with inclined upper chords and horizontal lower chords are generally the most desirable type for bridges. The exact analysis of this form is more complicated than the case of the symmetrical chords. However, by making a few reasonable assumptions a satisfactory solution may be obtained. Fig. 5(*a*) represents a Vierendeel truss with an inclined upper chord. Consider a typical panel, *a–b*, removed from the truss and cut into two parts by sections through the top of each vertical. The forces acting on each part are shown in Fig. 5(*b*) and Fig. 5(*c*).

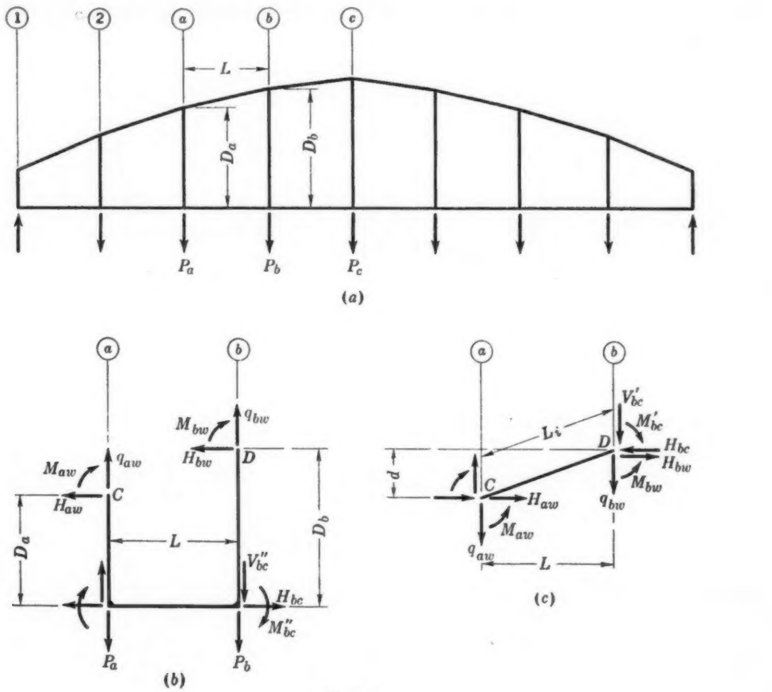

FIG. 5.

Considering the lower part, as shown in Fig. 5(*b*), the horizontal displacement of Point D with respect to Point C is:

$$E\,\Delta''_x = E\,\theta_C\,(D_b - D_a) + \int \frac{M\,y}{I}\,ds = E\,\theta_C\,(D_b - D_a)$$

$$+ \frac{1}{I_b}\int_0^{D_b} (M_{bw} - H_{bw}\,y)\,y\,dy + \frac{1}{I''_C}\int_0^{L}\Big[M_{bw} + M''_{bc} + (V''_{bc} + P_b$$

$$- q_{bw})\,x - H_{bw}\,D_b\Big]\,D_b\,dx + \frac{1}{I_a}\int_0^{D_a}\Big[H_{aw}\,(D_a - y) - M_{aw}\Big]\,(D_b - y)\,dy \;\ldots(21)$$

The value of the first integral in Equation (21) is:

$$\int_0^{D_b} (M_{bw} - H_{bw}\,y)\,y\,dy = M_{bw}\,\frac{D^2_b}{2} - H_{bw}\,\frac{D^3_b}{3}\;\ldots\ldots\ldots(22)$$

From statics, it is evident that (see Fig. 5(*b*) and Fig. 6(*b*)):

$$M''_{ba} = H_{bw}\,D_b - M_{bw} - M''_{bc}\ldots\ldots\ldots\ldots\ldots(23a)$$

and,

$$V''_{ab} = V''_{bc} + P_b - q_{bw}\ldots\ldots\ldots\ldots\ldots\ldots(23b)$$

Making these substitutions in the second integral of Equation (21) and integrating:

$$\int_0^L (V''_{ab}\, x - M''_{ba})\, D_b\, dx = V''_{ab}\, \frac{D_b L^2}{2} - M''_{ba}\, D_b\, L \quad \ldots\ldots (24)$$

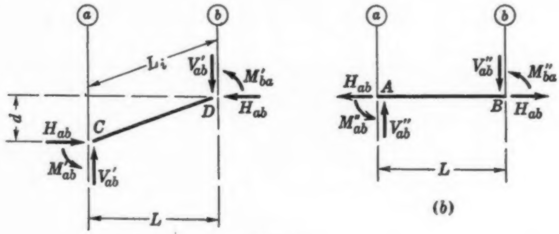

FIG. 6.

The value of the third integral is:

$$\int_0^{D_a} [H_{aw}(D_a - y) - M_{aw}]\,(D_b - y)\, dy = H_{aw}\, \frac{D_a^2}{6}\,(3\, D_b - D_a)$$

$$- M_{aw}\, \frac{D_a}{2}\,(2\, D_b - D_a) \quad \ldots\ldots (25)$$

Equations (22), (24), and (25), substituted in Equation (21) give:

$$E\, \Delta''_x = E\, \theta_C\,(D_b - D_a) - \frac{D_b^3}{3\, I_b}\, H_{bw} + \frac{D_b^2}{2\, I_b}\, M_{bw} + \frac{L^2\, D_b}{2\, I''_C}\, V''_{ab} - \frac{D_b\, L}{I''_C}\, M''_{ba}$$

$$+ \frac{D_a^2}{6\, I_a}\,(3\, D_b - D_a)\, H_{aw} - \frac{D_a}{2\, I_a}\,(2\, D_b - D_a)\, M_{aw} \quad \ldots\ldots (26)$$

The forces acting on the upper chord, as shown in Fig. 5(*c*), may be replaced by the equivalent system shown in Fig. 6(*a*). Then, for the upper chord, the horizontal displacement of Point *D* with respect to Point *C* is:

$$E\, \Delta'_x = E\, \theta_C\,(D_b - D_a) + \int \frac{M\, y}{I}\, ds = E\, \theta_C\,(D_b - D_a)$$

$$+ \frac{1}{I'_C} \int_0^{L_t} (V'_{ab}\, x - H_{ab}\, y - M'_{ba})\, y\, dL_t \quad \ldots\ldots (27)$$

From the geometry of the truss:

$$x = \frac{L}{d}\, y \quad \ldots\ldots (28a)$$

and,

$$d\, L_t = \frac{L_t}{d}\, dy \quad \ldots\ldots (28b)$$

Substituting Equations (28) in the integral of Equation (27), changing the limits of integration, and integrating:

$$\int_0^{L_t} (V'_{ab}\, x - H_{ab}\, y - M'_{ba})\, y\, dL_t = V'_{ab}\, \frac{L\, L_t\, d}{3} - H_{ab}\, \frac{L_t\, d^2}{3} - M'_{ba}\, \frac{L_t\, d}{2} \quad \ldots (29)$$

Substituting $d = D_b - D_a$ and $H_{ab} = \overset{a}{\underset{1}{\Sigma}} H_w$, Equation (29) becomes:

$$V'_{ab} \frac{L\, L_t}{3} (D_b - D_a) - \sum_1^a H_w \frac{L_t}{3} (D_b - D_a)^2 - M'_{ba} \frac{L_t}{2} (D_b - D_a) \text{ .(30)}$$

Equation (30) substituted in Equation (27) gives:

$$E\,\Delta'_x = E\,\theta_C (D_b - D_a) + \frac{L_t L}{3\, I'_C} (D_b - D_a)\, V'_{ab} - \frac{L_t}{2\, I'_C} (D_b - D_a)\, M'_{ba}$$

$$- \frac{L_t}{3\, I'_C} (D_b - D_a)^2 \sum_1^a H_w \text{..................(31)}$$

Equating Equations (26) and (31):

$$0 = - \frac{D^3_b}{3\, I_b} H_{bw} + \frac{D^2_a}{6 I_a} (3\, D_b - D_a)\, H_{aw} + \frac{D^2_b}{2\, I_b} M_{bw} - \frac{D_a}{2\, I_a} (2\, D_b - D_a)\, M_{aw}$$

$$+ \frac{L^2 D_b}{2\, I''_C} V''_{ab} - \frac{L\, L_t}{3\, I'_C} (D_b - D_a)\, V'_{ab} - \frac{D_b L}{I''_C} M''_{ba} + \frac{L_t}{2\, I'_C} (D_b - D_c)\, M'_{ba}$$

$$+ \frac{L_t}{3\, I'_C} (D_b - D_a)^2 \sum_1^a H_w \text{..................(32)}$$

Referring to Fig. 5(*b*), and Fig. 6(*b*) (which show the forces acting on the lower chord), the vertical deflection of Point D with respect to Point C is:

$$E\,\Delta''_y = - E\,\theta_C L + \frac{1}{I''_C} \int_0^L (M''_{ba} - V''_{ab}\, x)\, x\, dx$$

$$+ \frac{1}{I_a} \int_0^{D_a} (M_{aw} - H_{aw}\, y)\, L\, dy \text{ ....(33)}$$

Integrating Equation (33):

$$E\,\Delta''_y = - E\,\theta_C L + \frac{L^2}{2\, I''_C} M''_{ba} - \frac{L^3}{3\, I''_C} V''_{ab} + \frac{L\, D_a}{I_a} M_{aw} - \frac{L\, D^2_a}{2\, I_a} H_{aw} \text{ ..(34)}$$

Referring to Fig. 6(*a*), the vertical deflection of Point D with respect to Point C is:

$$E\,\Delta'_y = - E\,\theta_C L + \frac{1}{I'_C} \int_0^{L_t} (M'_{ba} - V'_{ab}\, x + H_{ab}\, y)\, x\, dL_t \text{ ....(35)}$$

Integrating Equation (35):

$$E\,\Delta'_y = - E\,\theta_C L + \frac{L\, L_t}{2\, I'_C} M'_{ba} - \frac{L^2 L_t}{3\, I'_C} V'_{ab} + \frac{L\, L_t}{3\, I'_C} (D_b - D_a)\, H_{ab} \text{..(36)}$$

Equating Equations (34) and (36):

$$0 = - \frac{L\, D^2_a}{2\, I_a} H_{aw} + \frac{L\, D_a}{I_a} M_{aw} - \frac{L^3}{3\, I''_C} V''_{ab} + \frac{L^2 L_t}{3 I'_C} V'_{ab} + \frac{L^2}{2\, I''_C} M''_{ba}$$

$$- \frac{L\, L_t}{2 I'_C} M'_{ba} - \frac{L\, L_t}{3\, I'_C} (D_b - D_a) \sum_1^a H_w \text{..............(37)}$$

Referring to Fig. 5(*b*), the angular displacement of Point D with respect to Point C is:

$$E\,\Delta''\,\theta = \int \frac{M}{I}\,ds = \frac{1}{I_b}\int_0^{D_b} (M_{bw} - H_{bw}\,y)\,dy + \frac{1}{I''_C}\int_0^{L} (V''_{ab}\,x - M''_{ba})\,dx + \frac{1}{I_a}\int_0^{D_a} (H_{aw}\,y - M_{aw})\,dy \quad \ldots\ldots(38)$$

Integrating Equation (38):

$$E\,\Delta''\,\theta = \frac{D_b}{I_b} M_{bw} - \frac{D^2_b}{2\,I_b} H_{bw} + \frac{L^2}{2\,I''_C} V''_{ab} - \frac{L}{I''_C} M''_{ba} + \frac{D^2_a}{2 I_a} H_{aw} - \frac{D_a}{I_a} M_{aw} \quad (39)$$

Referring to Fig. 6(*a*), the angular displacement of Point D with respect to Point C is:

$$E\,\Delta'\,\theta = \int \frac{M}{I}\,ds = \frac{1}{I'_C}\int_0^{L_t} (V'_{ab}\,x - H_{ab}\,y - M'_{ba})\,dL_t \quad \ldots(40)$$

Integrating Equation (40):

$$E\,\Delta'\,\theta = \frac{L\,L_t}{2\,I'_C} V'_{ab} - \frac{L_t}{2\,I'_C}(D_b - D_a)\,H_{ab} - \frac{L_t}{I'_C} M'_{ba} \quad \ldots\ldots(41)$$

Equating Equations (39) and (41):

$$0 = -\frac{D^2_b}{2\,I_b} H_{bw} + \frac{D^2_a}{2\,I_a} H_{aw} + \frac{D_b}{I_b} M_{bw} - \frac{D_a}{I_a} M_{aw} + \frac{L^2}{2\,I''_C} V''_{ab} - \frac{L\,L_t}{2\,I'_C} V'_{ab} - \frac{L}{I''_C} M''_{ba} + \frac{L_t}{I'_C} M'_{ba} + \frac{L_t}{2\,I'_C}(D_b - D_a)\sum_1^a H_w \quad \ldots\ldots(42)$$

Equations (32), (37), and (42) are the fundamental expressions for a typical panel. A set of these formulas could be written for each panel and the series solved simultaneously to find the unknowns. Such a procedure, although possible, would be too laborious for practical use. However, by transforming these fundamental equations, as described subsequently, a workable solution may be obtained.

Solution for the H_w-Values.—First, the terms involving M_{aw} and M_{bw} may be eliminated from Equations (32), (37), and (42). This can be accomplished by multiplying Equation (37) by $\frac{1}{2\,L}(D_a - D_b)$; thus:

$$0 = -\left(\frac{D^3_a}{4} - \frac{D^2_a\,D_b}{4}\right)\frac{H_{aw}}{I_a} + \left(\frac{D^2_a}{2} - \frac{D_a\,D_b}{2}\right)\frac{M_{aw}}{I_a} - \left(\frac{L^2\,D_a}{6} - \frac{L^2\,D_b}{6}\right)\frac{V''_{ab}}{I''_C} + \left(\frac{L L_t\,D_a}{6} - \frac{L\,L_t\,D_b}{6}\right)\frac{V'_{ab}}{I'_C} + \left(\frac{L\,D_a}{4} - \frac{L\,D_b}{4}\right)\frac{M''_{ba}}{I''_C} - \left(\frac{L_t\,D_a}{4} - \frac{L_t\,D_b}{4}\right)\frac{M'_{ba}}{I'_C} - \frac{L_t}{6\,I'_C}(D_b - D_a)(D_a - D_b)\sum_1^a H_w \quad \ldots\ldots(43)$$

Multiplying Equation (42) by $\frac{D_b}{2}$:

$$0 = -\frac{D^3_b}{4 I_b} H_{bw} + \frac{D_b D^2_a}{4 I_a} H_{aw} + \frac{D^2_b}{2 I_b} M_{bw} - \frac{D_a D_b}{2 I_a} M_{aw} + \frac{L^2 D_b}{4 I''_C} V''_{ab}$$

$$-\frac{L L_i D_b}{4 I'_C} V'_{ab} - \frac{L D_b}{2 I''_C} M''_{ba} + \frac{L_i D_b}{2 I'_C} M'_{ba} + \frac{L_i}{I'_C} (D_b - D_a) \frac{D_b}{4} \sum_1^a H_w \text{ . (44)}$$

Subtracting the sum of Equation (43) and (44) from Equation (32) gives Equation (45), as follows:

$$0 = -\frac{D^3_b}{12 I_b} H_{bw} + \frac{D^3_a}{12 I_a} H_{aw} + \frac{L^2}{12 I''_C} (D_b + 2 D_a) V''_{ab} + \frac{L L_i}{12 I'_C} (D_b + 2 D_a) V'_{ab}$$

$$- \frac{L}{4 I''_C} (D_b + D_a) M''_{ba} - \frac{L_i}{4 I'_C} (D_b + D_a) M'_{ba}$$

$$+ \frac{L_i}{12 I'_C} (2 D^2_a - D^2_b - D_a D_b) \sum_1^a H_w \text{ ....(45)}$$

Equation (45) may be simplified by substituting various statical relations. For equilibrium of the upper chord (see Fig. 6(*a*)), it is seen that:

$$V'_{ab} L - H_{ab} d = M'_{ab} + M'_{ba} \text{ ....(46a)}$$

For equilibrium of the lower chord (see Fig. 6(*b*)), it is seen that:

$$V''_{ab} L = M''_{ab} + M''_{ba} \text{ ....(46b)}$$

Let $\frac{M'_{ab} + M'_{ba}}{M''_{ab} + M''_{ba}} = \alpha$ and note that $H_{ab} = \sum_1^a H_w$; then, dividing Equation (46*a*) by Equation (46*b*):

$$V'_{ab} = \frac{d}{L} \sum_1^a H_w + \alpha V''_{ab} \text{ ....(47)}$$

Substituting $V'_{ab} = V_{ab} - V''_{ab}$ and $V''_{ab} = V_{ab} - V'_{ab}$ in Equation (47):

$$V'_{ab} = \frac{\alpha V_{ab}}{1 + \alpha} + \frac{\frac{d}{L} \sum_1^a H_w}{1 + \alpha} \text{ ....(48a)}$$

and,

$$V''_{ab} = \frac{V_{ab}}{1 + \alpha} - \frac{\frac{d}{L} \sum_1^a H_w}{1 + \alpha} \text{ ....(48b)}$$

Now, consider that part of the truss to the left of Panel Point b, as shown in Fig. 7. For equilibrium it is obvious that:

$$M_b - M'_{ba} - M''_{ba} - H_{ab}\, D_b = 0 \qquad (49)$$

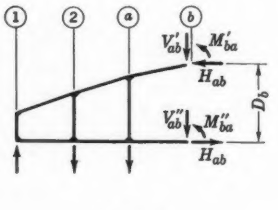

FIG. 7.

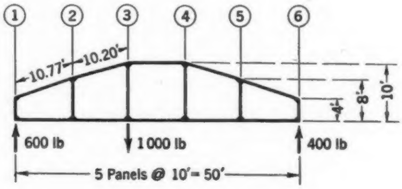

FIG. 8.

Let $\dfrac{M'_{ba}}{M''_{ba}} = \beta$ and note that $H_{ab} = \sum\limits_{1}^{a} H_w$; then, solving Equation (49), first for M'_{ba} and then for M''_{ba}:

$$M'_{ba} = \frac{\beta\, M_b}{1 + \beta} - \frac{\beta\, D_b \sum\limits_{1}^{a} H_w}{1 + \beta} \qquad (50a)$$

and,

$$M''_{ba} = \frac{M_b}{1 + \beta} - \frac{D_b \sum\limits_{1}^{a} H_w}{1 + \beta} \qquad (50b)$$

For reference purposes, designate the seven terms in Equation (45) by the letters, a to g, in order, from left to right. Substituting Equation (48b) in Term (c) gives:

$$\frac{L^2}{12\, I''_C}\,(D_b + 2\, D_a)\, V''_{ab} = \frac{L^2\,(D_b + 2\, D_a)}{12\, I''_C\,(1 + \alpha)}\, V_{ab} - \frac{L\,(D_b + 2\, D_a)\,(D_b - D_a)}{12\, I''_C\,(1 + \alpha)} \sum_{1}^{a} H_w \qquad (51c)$$

Substituting Equation (48a) in Term (d), gives:

$$\frac{L\, L_t\,(D_b + 2\, D_a)}{12\, I'_C}\, V'_{ab} = \frac{\alpha\, L\, L_i\,(D_b + 2\, D_a)}{12\, I'_C\,(1 + \alpha)}\, V_{ab} + \frac{L_t\,(D_b + 2\, D_a)\,(D_b - D_a)}{12\, I'_C\,(1 + \alpha)} \sum_{1}^{a} H_w \qquad (51d)$$

Substituting Equation (50b) in Term (e) gives:

$$- \frac{L}{4\, I''_C} (D_b + D_a)\, M''_{ba} = - \frac{L\,(D_b + D_a)}{4\, I''_C\,(1 + \beta)}\, M_b$$

$$+ \frac{L\, D_b\,(D_b + D_a)}{4\, I''_C\,(1 + \beta)} \sum_1^a H_w \ldots\ldots\ldots\ldots\ldots\ldots (51e)$$

Substituting Equation (50a) in Term (f) gives:

$$- \frac{L_t}{4\, I'_C} (D_b + D_a)\, M'_{ba} = - \frac{\beta\, L_t\,(D_b + D_a)}{4\, I'_C\,(1 + \beta)}\, M_b$$

$$+ \frac{\beta\, L_t\, D_b\,(D_b + D_a)}{4\, I'_C\,(1 + \beta)} \sum_1^a H_w \ldots\ldots\ldots\ldots\ldots (51f)$$

From Equations (51c) to (51f), collect the terms containing V_{ab}, to form Term (h):

$$\frac{L^2\,(D_b + 2\, D_a)}{12\, I''_C\,(1 + \alpha)}\, V_{ab} + \frac{\alpha\, L\, L_t\,(D_b + 2\, D_a)}{12\, I'_C\,(1 + \alpha)}\, V_{ab}$$

$$= \frac{L^2\,(D_b + 2\, D_a)}{12\, I''_C} \left[\frac{1}{1 + \alpha} + \frac{\dfrac{L_t\, I''_C}{L\, I'_C}\, \alpha}{1 + \alpha} \right] V_{ab}$$

$$= \frac{L^2\,(D_b + 2\, D_a)}{12\, I''_C} \left(\frac{1 + \phi\, \alpha}{1 + \alpha} \right) V_{ab} \ldots\ldots\ldots\ldots\ldots (51h)$$

in which ϕ = ratio of stiffness, top and bottom chords $= \dfrac{L_t\, I''_C}{L\, I'_C}$.

From the same group of terms, collect the ones containing M_b; thus, for Term (i):

$$- \frac{L}{4\, I''_C} \frac{(D_b + D_a)}{(1 + \beta)}\, M_b - \frac{\beta\, L_t\,(D_b + D_a)}{4\, I'_C\,(1 + \beta)}\, M_b$$

$$= - \frac{L}{4\, I''_C} (D_b + D_a) \left[\frac{1}{1 + \beta} + \frac{\dfrac{L_t\, I''_C}{L\, I'_C}\, \beta}{1 + \beta} \right] M_b$$

$$= - \frac{L}{4\, I''_C} (D_b + D_a) \left(\frac{1 + \phi\, \beta}{1 + \beta} \right) M_b \ldots\ldots\ldots\ldots\ldots (51i)$$

From the same group of terms, collect the ones containing $\sum_1^a H_w$, for Term (j):

$$- \frac{L\,(D^2_b + D_b\, D_a - 2\, D^2_a)}{12\, I''_C\,(1 + \alpha)} \sum_1^a H_w + \frac{L_t\,(D^2_b + D_b\, D_a - 2\, D^2_a)}{12\, I'_C\,(1 + \alpha)} \sum_1^a H_w$$

$$+ \frac{L\, D_b\,(D_b + D_a)}{4\, I''_C\,(1 + \beta)} \sum_1^a H_w + \frac{\beta\, L_t\, D_b\,(D_b + D_a)}{4\, I'_C\,(1 + \beta)} \sum_1^a H_w$$

$$= \left[\frac{L\,(2\, D^2_a - D_b\, D_a - D^2_b)}{12\, I''_C} \left(\frac{1 - \phi}{1 + \alpha} \right) + \frac{L\, D_b\,(D_b + D_a)}{4\, I''_C} \left(\frac{1 + \phi\, \beta}{1 + \beta} \right) \right] \sum_1^a H_w . (51j)$$

Adding Term (j) to Term (g), Term (k) equals:

$$\text{Term } (j) + \frac{L_t}{12\, I'_C}\,(2\, D^2_a - D_a\, D_b - D^2_b) \sum_1^a H_w$$

$$= \text{Term } (j) + \frac{\phi\, L}{12\, I''_C}\,(2\, D^2_a - D_a\, D_b - D^2_b) \sum_1^a H_w$$

$$= \left[\frac{L\,(2\, D^2_a - D_a\, D_b - D^2_b)}{12\, I''_C} \left(\frac{1 + \phi\, \alpha}{1 + \alpha}\right) + \frac{L\, D_b\,(D_b + D_a)}{4\, I''_C} \left(\frac{1 + \phi\, \beta}{1 + \beta}\right) \right] \sum_1^a H_w \quad \text{..........}(51k)$$

By these transformations, Terms (c) + (d) + (e) + (f) + (g) have been reduced to the three terms, (h) + (i) + (k). With these changes, Equation (45) becomes:

$$0 = -\frac{D^3_b}{12\, I_b}\, H_{bw} + \frac{D^3_a}{12\, I_a}\, H_{aw} + \frac{L}{12\, I''_C} \left[(2\, D^2_a - D_a\, D_b - D^2_b) \left(\frac{1 + \phi\, \alpha}{1 + \alpha}\right) + 3\, D_b\,(D_b + D_a) \left(\frac{1 + \phi\, \beta}{1 + \beta}\right) \right] \sum_1^a H_w - \frac{L}{4\, I''_C}\,(D_b + D_a) \left(\frac{1 + \phi\, \beta}{1 + \beta}\right) M_b + \frac{L^2}{12\, I''_C}\,(D_b + 2\, D_a) \left(\frac{1 + \phi\, \alpha}{1 + \alpha}\right) V_{ab} \quad \text{..........}(52)$$

Transposing the term containing H_{bw} and dividing through by its coefficient gives:

$$H_{bw} = \frac{D^3_a}{D^3_b}\,\frac{I_b}{I_a}\, H_{aw} + \frac{L}{D^3_b}\,\frac{I_b}{I''_C} \left[3\, D_b\,(D_b + D_a) \left(\frac{1 + \phi\, \beta}{1 + \beta}\right) + (2\, D^2_a - D_a\, D_b - D^2_b) \left(\frac{1 + \phi\, \alpha}{1 + \alpha}\right) \right] \sum_1^a H_w - \frac{3\, L}{D^3_b}\,\frac{I_b}{I''_C}\,(D_b + D_a) \left(\frac{1 + \phi\, \beta}{1 + \beta}\right) M_b + \frac{L^2}{D^3_b}\,\frac{I_b}{I''_C}\,(D_b + 2\, D_a) \left(\frac{1 + \phi\, \alpha}{1 + \alpha}\right) V_{ab} \quad \text{..........}(53)$$

Equation (53) is the fundamental expression for determining the H_w-values. No approximations have been made in its derivation, except to neglect deflections due to axial stresses. The effect of these deflections is considered subsequently. For the case of parallel chords and constant moment of inertia, Equation (53) reduces to Equation (11). It may be used for parallel-chord trusses with members having different moments of inertia.

After evaluating the coefficients of each term of Equation (53), it reduces to the simple form of Equation (13). In this case, however, the coefficients depend not only on the truss dimensions, but also upon α and β, which are ratios of the bending moments in the upper chord to those in the lower chord.

In order to evaluate the coefficients it is necessary to assume values for these ratios. It can be shown that, in most cases, choosing widely different values for these ratios effects the H_w-values only slightly. Hence, by assuming any reasonable values for α and β, a close approximation to the true values of H_w is obtained upon solving the equation.

For the particular case in which the moments of inertia are such that $\phi = \frac{L_t}{L}\frac{I''_C}{I'_C} = 1$, the factors involving α and β reduce to unity and the solution becomes exact. For values of ϕ close to unity, the factors involving α and β will be nearly equal to unity for any values of these ratios and, therefore, in such case, little error is introduced in assuming these ratios.

The following examples illustrate the method of solution and the effect on the H_w-values of assuming different magnitudes for α and β.

Example 3.—Consider a truss with dimensions and loads as shown in Fig. 8. For this example assume that the moments of inertia are of such values that $I_a = I_b = I''_c = I'_c \frac{L}{L_t}$. Then, for this case, ϕ will equal 1, and Equation (53) reduces to:

$$H_{bw} = \frac{D^3_a}{D^3_b} H_{aw} + \frac{2\,L}{D^3_b} (D^2_a + D_a\,D_b + D^2_b) \sum_1^a H_w - \frac{3\,L}{D^3_b} (D_b + D_a)\,M_b + \frac{L^2}{D^3_b} (D_b + 2\,D_a)\,V_{ab} \quad (54)$$

Apply Equation (54) to each panel in the left half of the truss. Then, after evaluating the coefficients, the following series of equations is obtained:

$$H_{1w} = H_{1w} \quad (55a)$$

$$H_{2w} = 0.125\,H_{1w} + 4.37\,H_{1w} - 0.702\,M_2 + 3.12\,V_{12} \quad (55b)$$

$$H_{3w} = 0.512\,H_{2w} + 4.88\,(H_{1w} + H_{2w}) - 0.54\,M_3 + 2.60\,V_{23} \quad (55c)$$

and,

$$H_{4w} = 1.00\,H_{3w} + 6.00\,(H_{1w} + H_{2w} + H_{3w}) - 0.60\,M_4 + 3.00\,V_{34} \quad (55d)$$

For the loading given: $M_2 = 6\,000$ ft-lb; $M_3 = 12\,000$ ft-lb; $M_4 = 8\,000$ ft-lb; $V_{12} = 600$ lb; $V_{23} = 600$ lb; and, $V_{34} = -400$ lb. Substituting these values in Equations (55) and then expressing each value of H_w in terms of H_{1w}:

$$H_{1w} = H_{1w} \quad (56a)$$

$$H_{2w} = 4.495\,H_{1w} - 2\,340 \quad (56b)$$

$$H_{3w} = 0.512\,H_{2w} + 4.88\,(H_{1w} + H_{2w}) - 4\,920 = 29.117\,H_{1w} - 17\,537 \quad (56c)$$

and,

$$H_{4w} = 1.00\,H_{3w} + 6.00\,(H_{1w} + H_{2w} + H_{3w}) - 6\,000 = 236.789\,H_{1w} - 142\,799 \quad (56d)$$

Now consider the truss reversed in position, as shown in Fig. 9. The prime mark, ($'''$), will be used to designate terms for this position. Moments

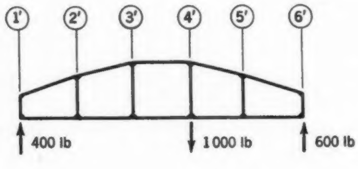

FIG. 9.

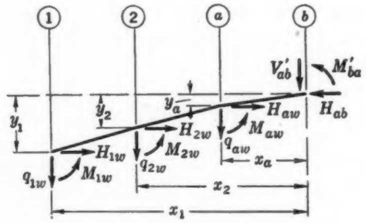

FIG. 10.

and shears for the left half of the truss are: $M'''_2 = 4\,000$ ft-lb; $M'''_3 = 8\,000$ ft-lb; $V'''_{12} = 400$ lb; and $V'''_{23} = 400$ lb. Substituting these values in Equations (55) and then expressing each value of H'''_w in terms of H'''_{1w}:

$$H'''_{1w} = H'''_{1w} \quad\text{...}(57a)$$

$$H'''_{2w} = 4.495\ H'''_{1w} - 1\,560 \quad\text{.................................}(57b)$$

and,

$$H'''_{3w} = 0.512\ H'''_{2w} + 4.88\ (H'''_{1w} + H'''_{2w}) - 3\,280$$

$$= 29.117\ H'''_{1w} - 11\,692 \quad\text{.................}(57c)$$

Since it is obvious from their identity that $H_{4w} = -\ H'''_{3w}$: $236.789\ H_{1w} - 142\,799 = -\ 29.117\ H'''_{1w} + 11\,692$; and,

$$H'''_{1w} = -\ 8.13233\ H_{1w} + 5\,305.87 \quad\text{..............}(58)$$

For equilibrium:

$$H_{1w} + H_{2w} + H_{3w} + H_{4w} - H'''_{2w} - H'''_{1w} = 0 \quad\text{.......}(59)$$

Substituting for each value of H_w in Equation (59), its value in terms of H_{1w}: $271.401\ H_{1w} - 162\,676 - 5.495\ H'''_{1w} + 1\,560 = 0$; $271\,401\ H_{1w} - 162\,676 - 5.495\ (-\ 8.13233\ H_{1w} + 5\,305.87) + 1\,560 = 0$; and, $H_{1w} = 601.96$. Substituting this value for H_{1w} in Equations (56), (57), and (58): $H_{2w} = 366$ lb; $H_{3w} = -\ 10$ lb; $H_{4w} = -\ 262$ lb; $H_{5w} = -\ H'''_{2w} = -\ 285$ lb; and, $H_{6w} = -\ H'''_{1w} = -\ 411$ lb.

Example 4.—Consider the same truss as in Example 3, but assume that $I_a = I_b = I''_c = I'_c$. In this case, in order to evaluate the coefficients, it is necessary to assume values for α and β. To show how slightly these values affect the magnitudes of the H_w, it will be assumed that $\alpha = \beta = 2$. This is a highly improbable value, of course, and is chosen only for purposes of illustration. A more reasonable assumption would give more accurate results.

Referring to Equation (53), the factors, $\frac{1+\phi\,\alpha}{1+\alpha}$ and $\frac{1+\phi\,\beta}{1+\beta}$, become $\dfrac{1+\dfrac{10.77\times 2}{10}}{1+2} = 1.051$ for the first panel; $\dfrac{1+\dfrac{10.2\times 2}{10}}{1+2} = 1.013$ for the second panel; and $\dfrac{1+\dfrac{10\times 2}{10}}{1+2} = 1$ for the third panel. The coefficients for this example will equal those in Example 3 multiplied by these factors, except for the H_{aw}-term, which remains the same. Performing this multiplication yields the following series of equations:

$$H_{1w} = H_{1w} \qquad \text{(60a)}$$

$$H_{2w} = 0.125\, H_{1w} + 4.59\, H_{1w} - 0.738\, M_2 + 3.28\, V_{12} \qquad \text{(60b)}$$

$$H_{3w} = 0.512\, H_{2w} + 4.94\, (H_{1w} + H_{2w}) - 0.547\, M_3 + 2.63\, V_{23} \qquad \text{(60c)}$$

and,

$$H_{4w} = 1.00\, H_{3w} + 6.00\, (H_{1w} + H_{2w} + H_{3w}) - 0.600\, M_4 + 3.00\, V_{34} \qquad \text{(60d)}$$

Substituting the values for the moments and shears and solving as was done in Example 3: $H_{1w} = 600$; $H_{2w} = 369$; $H_{3w} = -10$; $H_{4w} = -262$; $H_{5w} = -H'''_{2w} = -287$; and, $H_{6w} = H'''_{1w} = -409$. Comparing these values with those of Example 3, it is seen that the two assumptions give nearly the same results, even though α and β in this example were chosen far greater than their probable value.

Determination of Chord Moments and Shears

After the H_w-values have been determined from Equation (53), the next step is to solve for the M_w and q_w-values. This is done by transforming the fundamental equations, as follows: Multiplying Equation (37) by $\frac{3}{L}$ and adding the result to Equation (42), gives:

$$-\frac{D^2_b}{2\,I_b} H_{bw} - \frac{D^2_a}{I_a} H_{aw} + \frac{D_b}{I_b} M_{bw} + \frac{2\,D_a}{I_a} M_{aw} - \frac{L^2}{2\,I''_C} V''_{ab} + \frac{L\,L_1}{2\,I'_C} V'_{ab}$$
$$+ \frac{L}{2\,I''_C} M''_{ba} - \frac{L_1}{2\,I'_C} M'_{ba} - \frac{L_1}{2\,I'_C} (D_b - D_a) \sum_1^a H_w = 0 \qquad \text{(61)}$$

Substituting the statical relations, $V''_{ab} = V_{ab} - V'_{ab}$ and $M''_{ba} = M_b - M'_{ba} - D_b \sum\limits_1^a H_w$, in Equation (61) and collecting terms:

$$-\frac{D^2_b}{2\,I_b} H_{bw} + \frac{D^2_a}{I_a} H_{aw} + \frac{D_b}{I_b} M_{bw} + \frac{2\,D_a}{I_a} M_{aw} - \frac{L^2}{2\,I''_C} V_{ab} + \frac{L^2}{2\,I''_C} (1+\phi)\, V'_{ab}$$
$$+ \frac{L}{2\,I''_C} M_b - \frac{L}{2\,I''_C} (1+\phi)\, M'_{ba} + \left[\frac{L_1\,D_a}{2\,I''_C} - \frac{L\;D_b}{2\,I''_C}(1+\phi)\right] \sum_1^a H_w = 0 \qquad \text{(62)}$$

Fig. 10 shows the forces acting on the upper chord to the left of Panel Point b. For equilibrium:

$$M'_{ba} = -\sum_1^a M_w - \sum_1^a (q_w\,x) - \sum_1^a (H_w\,y) = -\sum_1^a M_w$$
$$+ \sum_1^a (V'\,L) - \sum_1^a (H_w\,y) \ldots\ldots\ldots\ldots\ldots\ldots\ldots(63)$$

Equation (63) may be written:

$$M'_{ba} = -M_{aw} - \sum_1^{(a-1)} M_w + V'_{ab}\,L + \sum_1^{(a-1)} (V'\,L) - \sum_1^a (H_w\,y) \quad \ldots(64)$$

From the geometry of the truss it is seen that $H_w\,y = H_w\,D_b - H_w\,D$. Substituting this value in Equation (64):

$$M'_{ba} = -M_{aw} - \sum_1^{(a-1)} M_w + V'_{ab}\,L + \sum_1^{(a-1)} (V'\,L)$$
$$- \sum_1^a (H_w\,D_b) + \sum_1^a (H_w\,D) \ldots\ldots\ldots\ldots\ldots\ldots(65)$$

Substituting Equation (65) in Equation (62) and collecting terms:

$$\frac{D_b}{I_b}\,M_{bw} = -\left[\frac{2\,D_a}{I_a} + \frac{L}{2\,I''_C}\,(1+\phi)\right] M_{aw} - \frac{L}{2\,I''_C}\,(1+\phi) \sum_1^{a-1} M_w$$
$$+ \frac{L^2}{2\,I''_C}\,(1+\phi) \sum_1^{a-1} V' + \frac{D^2_b}{2\,I_b}\,H_{bw} + \frac{D^2_a}{I_a}\,H_{aw} + \frac{L^2}{2\,I''_C}\,V_{ab} - \frac{L}{2\,I''_C}\,M_b$$
$$- \frac{L_t\,D_a}{2\,I'_C} \sum_1^a H_w + \frac{L}{2\,I''_C}\,(1+\phi) \sum_1^a (H_w\,D) \quad \ldots\ldots\ldots(66)$$

For the first panel Equation (66) reduces to:

$$\frac{D_2}{I_b}\,M_{2w} = -\left[\frac{2\,D_1}{I_a} + \frac{L}{2I''_C}\,(1+\phi)\right] M_{1w} + \left[\frac{D^2_1}{I_a} + \frac{L\;D_1}{2\,I''_C}\right] H_{1w} + \frac{D^2_2}{2\,I_b}\,H_{2w} \ldots(67)$$

Substituting the same statical relations in Equation (37) and simplifying:

$$\frac{L^2}{6\,I''_C}\,(1+\phi)\,V'_{ab} = \left[\frac{D_a}{I_a} + \frac{L}{2\,I''_C}\,(1+\phi)\right] M_{aw} + \frac{L}{2\,I''_C}\,(1+\phi) \sum_1^{a-1} M_w$$
$$- \frac{L^2}{2\,I''_C}\,(1+\phi) \sum_1^{a-1} V' - \frac{D^2_a}{2\,I_a}\,H_{aw} - \frac{L^2}{3\,I''_C}\,V_{ab} + \frac{L}{2\,I''_C}\,M_b$$
$$+ \frac{L_t}{6\,I'_C}\,(D_b + 2\,D_a) \sum_1^a H_w - \frac{L}{2\,I''_C}\,(1+\phi) \sum_1^a (H_w\,D) \;\ldots.(68)$$

For the first panel Equation (68) reduces to:

$$\frac{L^2}{6\,I''_C}\,(1+\phi)\,V'_{12} = \left[\frac{D_1}{I_a} + \frac{L}{2\,I''_C}\,(1+\phi)\right] M_{1w}$$

$$-\left[\frac{D^2_1}{2\,I_a} - \frac{L_t}{6\,I'_C}\,(D_2 + 2\,D_1) + \frac{L\,D_1}{2\,I''_C}\,(1+\phi)\right] H_{1w} + \frac{L^2}{6\,I''_C}\,V_{12} \quad \ldots\ldots (69)$$

Equations (66), (67), (68), and (69) furnish the necessary relations for solving for each value of M_w and V'. Each value of q_w, of course, is determined directly from the V'-values.

Equations (66) to (69) appear somewhat involved but, after substituting numerical values for the truss dimensions and for the previously calculated H_w-values, they reduce to:

$$C_1\,M_{bw} = C_2\,M_{aw} + C_3 \sum_{1}^{a-1} M_w + C_4 \sum_{1}^{a-1} V' + C_5 \quad \ldots\ldots\ldots (70)$$

which corresponds to Equation (66);

$$C_6\,M_{2w} = C_7\,M_{1w} + C_8 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (71)$$

which corresponds to Equation (67);

$$C_9\,V'_{ab} = C_{10}\,M_{aw} + C_{11} \sum_{1}^{a-1} M_w + C_{12} \sum_{1}^{a-1} V' + C_{13} \quad \ldots\ldots (72)$$

which corresponds to Equation (68); and,

$$C_{14}\,V'_{12} = C_{15}\,M_{1w} + C_{16} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (73)$$

which corresponds to Equation (69).

The general method of solution is to apply Equations (66) and (68) to each panel in turn. This gives a series of expressions which may be solved simultaneously to obtain the magnitude of each unknown function. It will be noted that the equations for any given panel involve the moments, M_w, and shears, V', for the preceding panels only. Hence, by starting with the equations for the end panel and substituting, successively, in the equations for the following panels, each M_w may be expressed in terms of M_{1w} and the entire series is then readily solved.

This method of solution is best explained by solving an example. Accordingly, Equations (66) to (69) will be applied to find the values of M_w and V' for the truss used in Example 3.

Example 5.—For this example, the moments and shears will be computed for the truss of Example 3, using the H_w-values obtained in that example. For this case it should be noted that $\phi = 1$ and that $\frac{L_t}{I'_C} = \frac{L}{I''_C}$. Then, for this

particular problem, Equations (66) to (69) become:

For Equation (66):

$$D_b\,M_{bw} = -\,(2\,D_a + L)\,M_{aw} - L\sum_1^{a-1} M_w + L^2\sum_1^{a-1} V' + \frac{D^2_b}{2}\,H_{bw}$$

$$+\,D^2_a\,H_{aw} + \frac{L^2}{2}\,V_{ab} - \frac{L}{2}\,M_b - \frac{L\,D_a}{2}\sum_1^{a} H_w + L\sum_1^{a}(H_w\,D)\;..(74)$$

For Equation (67):

$$D_2\,M_{2w} = -\,(2\,D_1 + L)\,M_{1w} + \left(D^2_1 + \frac{L\,D_1}{2}\right)H_{1w} + \frac{D^2_2}{2}\,H_{2w}\;...(75)$$

For Equation (68):

$$\frac{L^2}{3}\,V'_{ab} = (D_a + L)\,M_{aw} + L\sum_1^{a-1} M_w - L^2\sum_1^{a-1} V' - \frac{D^2_a}{2}\,H_{aw} - \frac{L^2}{3}\,V_{ab}$$

$$+\,\frac{L}{2}\,M_b + \frac{L}{6}\,(D_b + 2\,D_a)\sum_1^{a} H_w - L\sum_1^{a}(H_w\,D)\;......(76)$$

and, for Equation (69):

$$\frac{L^2}{3}\,V'_{12} = (D_1 + L)\,M_{1w} - \left[\frac{D^2_1}{2} - \frac{L}{6}\,(D_2 + 2\,D_1) + L\,D_1\right]H_{1w} + \frac{L^2}{6}\,V_{12}\,.(77)$$

Applying Equations (75) and (77) to the first panel gives:

$$8\,M_{2w} = -\,(2\times 4 + 10)\,M_{1w} + \left(4^2 + \frac{10\times 4}{2}\right)(602) + \frac{8^2}{2}\,(366);\text{ or,}$$

$$M_{2w} = 4\,173 - 2.25\,M_{1w}\;..................(78)$$

and, $\frac{10^2}{3}\,V'_{12} = (4 + 10)\,M_{1w} - \left[\frac{4^2}{2} - \frac{10}{6}\,(8 + 2\times 4) + 10\times 4\right](602)$ $+\,\frac{10}{6}\,(600)$; or,

$$V'_{12} = 0.420\,M_{1w} - 86\;....................(79)$$

Applying Equations (74) and (76) to the second panel gives:

$10\,M_{3w} = -\,(2\times 8 + 10)\,M_{2w} - 10\,M_{1w} + 10^2\,V'_{12} + \frac{10^2}{2}\,(-10)$ $+\,8^2\,(366) + \frac{10^2}{2}\,(600) - \frac{10}{2}\,(12\,000) - \frac{10\times 8}{2}\,(602 + 366) + 10\,(602\times 4$ $+\,366\times 8)$; or,

$$M_{3w} = -\,2.6\,M_{2w} - M_{1w} + 10\,V'_{12} + 759\;..........(80)$$

and, $\frac{10^2}{3} V'_{23} = (8 + 10)\ M_{2w} + 10\ M_{1w} - 10^2\ V'_{12} - \frac{8^2}{2}\,(366) - \frac{10^2}{3}\,(600)$
$+ \frac{10}{2}\,(12\,000) + \frac{10}{6}\,(10 + 2 \times 8)\ (602 + 366) - 10\ (602 \times 4 + 366 \times 8)$; or,

$$V'_{23} = 0.54\ M_{2w} + 0.3\ M_{1w} - V'_{12} + 508 \ldots\ldots\ldots\ldots (81)$$

Applying Equations (74) and (76) to the third panel gives: $10\ M_{4w} = -\ (2 \times 10 + 10)\ M_{3w} - 10\ (M_{2w} + M_{1w}) + 10^2\ (V'_{23} + V'_{12}) + \frac{10^2}{2}\,(-\,262) + 10^2\ (-\,10) + \frac{10^2}{2}\,(-\,400) - \frac{10}{2}\,(8\,000) - \frac{10 \times 10}{2}\ (602 + 366 - 10) + 10\,(602 \times 4 + 366 \times 8 - 10 \times 10)$; or,

$$M_{4w} = -\ 3\ M_{3w} - M_{2w} - M_{1w} + 10\ V'_{23} + 10\ V'_{12} - 6\,964 \ldots (82)$$

and, $\frac{10^2}{3}\ V'_{34} = (10 + 10)\ M_{3w} + 10\ (M_{2w} + M_{1w}) - 10^2\ (V'_{23} + V'_{12}) - \frac{10^2}{2}\,(-\ 10) - \frac{10^2}{3}\,(-\ 400) + \frac{10}{2}\,(8\,000) + \frac{10}{6}\,(602 + 366 - 10)\ (10 + 2 \times 10) - 10\ (602 \times 4 + 366 \times 8 - 10 \times 10)$; or,

$$V'_{34} = 0.6\ M_{3w} + 0.3\ M_{2w} + 0.3 M_{1w} - 3\ V'_{23} - 3\ V'_{12} + 1\,481 \ldots (83)$$

Substituting Equations (78) and (79) in Equations (80) and (81):

$$M_{3w} = 9.05\ M_{1w} - 10\,960 \ldots\ldots\ldots\ldots\ldots\ldots (84a)$$

and,

$$V'_{23} = 3\,021 - 2.175\ M_{1w} \ldots\ldots\ldots\ldots\ldots\ldots (84b)$$

Substituting Equations (78), (79), and (84), in Equations (83) and (82):

$$M_{4w} = 51\,090 - 43.45\ M_{1w} \ldots\ldots\ldots\ldots\ldots\ldots (85a)$$

and,

$$V'_{34} = 10.32\ M_{1w} - 12\,648 \ldots\ldots\ldots\ldots\ldots\ldots (85b)$$

By means of these substitutions, all the values of M_w and V' are expressed in terms of M_{1w}.

Now, consider the truss turned end for end as shown in Fig. 9, and write a corresponding series of expressions, beginning with the first panel on the left, as before. Functions for the truss in this reversed position will be designated by the prime mark ('″). The coefficients for all terms will be the same as for the truss in its original position. The only difference in the calculations is that the values of H_w, M_b, and V_{ab}, of course, must be those for the reversed position. Because of this similarity, the detailed calculations are

omitted in this case and only the final equations are given. Applying Equations (75) and (77) to the first panel, gives:

$$M'''_{2w} = 2\,990 - 2.25\ M'''_{1w} \dots\dots\dots\dots\dots\dots (86a)$$

and,

$$(V'_{12})''' = 0.42\ M'''_{1w} - 63.04 \dots\dots\dots\dots\dots\dots (86b)$$

Applying Equations (74) and (76) to the second panel:

$$M'''_{3w} = -\ 2.6\ M'''_{2w} - M'''_{1w} + 10\ (V'_{12})''' + 2\,274 \dots (87a)$$

and,

$$(V'_{23})''' = 0.54\ M'''_{2w} + 0.3\ M'''_{1w} - 3\ (V'_{12})''' + 245 \dots\dots (87b)$$

Applying Equations (74) and (76) to the third panel, gives:

$$M'''_{3w} = -3\ M'''_{3w} - M'''_{2w} - M'''_{1w} + 10\ (V'_{23})''' + 10\ (V'_{12})''' + 420. (88a)$$

and,

$$(V'_{34})''' = 0.6\ M'''_{3w} + 0.3\ M'''_{2w} + 0.3\ M'''_{1w} - 3\ (V'_{23})''' - 3\ (V'_{12})''' + 481 . (88b)$$

Substituting Equations (86) in Equations (87):

$$M'''_{3w} = 9.05\ M'''_{1w} - 6\,130 \dots\dots\dots\dots\dots\dots (89a)$$

and,

$$(V'_{23})''' = 2\,049 - 2.175\ M'''_{1w} \dots\dots\dots\dots\dots\dots (89b)$$

Substituting Equations (86) and (89) in Equations (88):

$$M'''_{4w} = 35\,680 - 43.45\ M'''_{1w} \dots\dots\dots\dots\dots\dots (90a)$$

and,

$$(V'_{34})''' = 10.32\ M'''_{1w} - 8\,258 \dots\dots\dots\dots\dots\dots (90b)$$

It is obvious that $M_{3w} = -\ M'''_{4w}$ and that $M_{4w} = -\ M'''_{3w}$. Equating the values for these functions as given in Equations (84), (85), (89*a*), and (90*a*):

$$9.05\ M_{1w} - 10\,960 = -\ 35\,680 + 43.45\ M'''_{1w} \dots\dots\dots (91a)$$

and,

$$51\,090 - 43.45\ M_{1w} = -\ 9.05\ M'''_{1w} + 6\,130 \dots\dots\dots (91b)$$

Solving Equations (91*a*) and (91*b*) the following values are obtained: $M_{1w} = 1\,206$; and, $M'''_{1w} = 820$. The remaining moments and shears may now be found by substituting these values in Equations (78), (79), (84), (85), (86), and (89). This gives the following results: $M_{2w} = 1\,460$; $M_{3w} = -\ 50$; $M_{4w} = -\ M'''_{3w} = -\ 1\,290$; $M_{5w} = -\ M'''_{2w} = -\ 1\,150$; $V'_{12} = 420$; $V'_{23} = 399$; $V'_{34} = -\ 205$; $V'_{45} = -\ (V'_{23})''' = -\ 265$; and, $V'_{56} = -\ (V'_{12})''' = -\ 281$.

The q_w-values are determined directly from the V'-values, giving: $q_{1w} = -420$; $q_{2w} = 21$; $q_{3w} = 604$; $q_{4w} = 60$; $q_{5w} = 16$; and $q_{6w} = -281$.

This completes the analysis of the truss. With the values of H_w, M_w, and V' known, the stresses at all points in the truss may be calculated.

Special Approximations

Only general methods of solution have been illustrated in Examples 1 to 5. There are various places where short-cut methods of computation or, in special cases, further simplifications may be applied. Furthermore, the general equations may be solved by successive approximations.

For example, the values of H_w may be obtained by successive approximations, as follows: Equations (10) and (53) are of the form shown in Equation (13). Assume the location of the points of contraflexure in Panel bc and calculate the value of H_{bc}. Then, substituting the relation, $H_{bw} = H_{bc} - \sum_1^a H_w$, into Equation (13) an expression of the form, $H_{aw} = C' \sum_1^{a-1} H_w + C''$, is obtained. Starting with an end panel, an equation of this form may be set up and solved for each panel in turn, since $\sum_1^{a-1} H_w$ is known from the preceding calculations. The operations may then be repeated using for H_{bc} the results of the first approximation. Similarly, Equations (66) and (68) may be solved by successive approximations by assuming a value for M_{bw}. A good assumption for a first approximation is $M_{bw} = \frac{1}{2} D_b H_{bw}$.

Professor Vierendeel uses an approximate method for calculating the chord moments and shears[12] which is very rapid. Equations (48) and (50) are expressions for these functions. After determining the values of the H_w-values from Equation (53), the only unknowns in these expressions are α and β. By assuming values for these ratios, the desired moments and shears may be obtained quickly. Professor Vierendeel uses the assumption that $\alpha = \beta = \dfrac{I'_C}{I''_C} \dfrac{L}{L_i}$. It is obvious that this assumption greatly simplifies the work in comparison with the method used in Example 5. Experience in analyzing many trusses has shown that this approximation gives results which are sufficiently accurate for most designs. The only serious errors that occur as a result of using this method are in the values for the moments and shears in one or two panels at the ends of some trusses. For such cases a close approximation may be secured by combining the general method presented in this paper with Vierendeel's method; that is, use the former to obtain the moments and shears in the first two panels and the latter for the remainder of the truss.

Effect of Axial Stress

The effect of deformation due to axial stress in the chord members upon the H_w-values may be taken into account by the addition of two terms

[12] "Cours de Stabilité des Constructions", Tome IV, by Arthur Vierendeel, Louvain. 1920.

to Equation (53). Refer to Equation (21) which is the expression for the horizontal deflection in the lower part of a typical panel. The deflection due to direct stress in the lower chord is obviously $\frac{H_{ab}\,L}{A''\,E} = \frac{L \sum_{1}^{a} H_w}{A''\,E}$. The total deflection is obtained by adding this term to Equation (21).

Similarly, for the upper chord, the horizontal deflection due to direct stress is $-\left(\frac{L\,H_{ab}}{A'\,E} + \frac{V'_{ab}\,d}{A'\,E}\right)\frac{L}{L_t}$. Substituting for V'_{ab} its value from Equation (48), these terms become:

$$-\left[\frac{L \sum_{1}^{a} H_w}{A'\,E} + \frac{\alpha\,(D_b - D_a)}{(1+\alpha)\,A'\,E}\,V_{ab} + \frac{(D_b - D_a)^2}{(1+\alpha)\,L\,A'\,E}\sum_{1}^{a} H_w\right]\frac{L}{L_t}$$

Adding these terms to Equation (27), the total horizontal deflection for the upper chord is obtained.

Adding these terms to Equations (21) and (27) and carrying them through all the operations to Equation (53), they become, in the final stage:

$$\frac{12\,I_b}{A'\,D^3_b}\left\{\sum_{1}^{a} H_w\left[\frac{L\,A'}{A''} + \frac{L^2}{L_t} + \frac{(D_b - D_a)^2}{L_t\,(1+\alpha)}\right] + V_{ab}\left[(D_b - D_a)\frac{L}{L_t}\left(\frac{\alpha}{1+\alpha}\right)\right]\right\}$$

By adding this expression to the right-hand side of Equation (53), the effect of direct stress is included in the solution. This is equivalent to slightly increasing the numerical values of the coefficients of the original terms, involving $\sum_{1}^{a} H_w$ and V_{ab}. In most cases, the change is small and may be neglected. In applying the foregoing term, it should be noted that I and A must be in the same units as D and L.

For trusses with symmetrical chords the effect of axial stresses may be included in a similar manner.

The deformation of the web members due to axial stress could be included in Equation (37), but this effect is generally so slight as to be negligible.

Conclusions

In summary, the following conclusions regarding this method of analyzing Vierendeel trusses and its limitations are noted:

(1) All the equations derived are based upon the principles of virtual work and are, therefore, "exact"; that is, they include no more assumptions or limitations than the basic elastic theory;

(2) The solution of the equations for trusses with symmetrical chords requires no assumptions and is, therefore, "exact";

(3) The solution of the equations for trusses with inclined upper chords involves an assumption of values for α and β, and, therefore, is approximate;

(4) The method of analysis presented in this paper, although not extremely rapid or simple, is reasonably so, and gives results that are sufficiently accurate for good design; and,

(5) A large part of the labor involved in the analysis is in the evaluation of the coefficients and setting up an equation for each panel. This part of the solution depends only upon the truss dimensions. Therefore, in investigating the effect of different loadings to obtain influence lines, this part of the analysis is the same for all cases, and so need be performed only once.

In any particular solution, the numerical effect of Assumption (3) upon the value of the coefficients is immediately evident and its effect upon the results may be estimated. In cases where the assumption has a large effect upon coefficient values, this method should be used with caution. In designing it is possible to assume such relative values for the moments of inertia of the members that the effect of the assumption is negligible or even nil.

ACKNOWLEDGMENTS

Grateful acknowledgment is due Professor Arthur Vierendeel for his kind assistance in investigating this form of truss and for furnishing much invaluable data regarding the bridges which he has designed.

APPENDIX I

NOTATION

The following symbols, defined where first introduced in the paper, are re-arranged herein for convenience of reference: In general, single primes denote upper chord values; double primes denote lower chord values; and triple primes denote values for the truss in a reversed position.

A = area; A' = cross-sectional area of an upper chord member; A'' = cross-sectional area of a lower chord member;

a = a subscript; Subscripts a, b, and c, denote three successive panel points (see Fig. 1), and are used to designate the location of the various functions; for example, V'_{ab} = vertical shear in the upper chord between Panel Points a and b; M''_{bc} = bending moment at Point b in the lower chord member, bc; that is, the member extending between Panel Points b and c; M_b = external moment at Point b; H_{aw} = shear in web member at Panel Point a;

b = a subscript (see a);

C = a constant; as a subscript to the moment of inertia, C denotes "chord member";

c = a subscript (see a);

D = depth of a truss at a given panel point (designated by subscript); length of a web member transverse to the longitudinal axis of a truss;

d = drop in chord between panel points; for trusses with only the upper chord inclined, $d = D_b - D_a$; for trusses with symmetrical chords, $d = \frac{1}{2}(D_b - D_a)$;

E = modulus of elasticity;

H = in general, a horizontal force; a horizontal component of axial compression or tension in a chord member; H_w = the total transverse shear in a web member; $\sum_1^a H_w$ = the sum of the H_w-values from Panel Point 1 to Panel Point a, inclusive; H_{aw} = the shear in a web member at Point a;

I = moment of inertia; I'_c and I''_c = moments of inertia of the upper and lower chord members, respectively; I_a and I_b = moments of inertia of the web members at Panel Points a and b, respectively;

i = a subscript denoting "inclined";

L = length of a panel; L_i = length of an inclined chord member;

M = external moment, considered positive when the external forces on that part of the truss to the left of the point of moments have a clockwise moment about that point. Internal moments and forces are considered positive when they act in the direction shown in the illustrations; M' and M'' = bending moments in the upper and lower chord members, respectively; M_{aw} = bending moment at Point C on the upper end of web member (at the panel point, a); M_{bw} = bending moment at Point D on the upper end of web member (at the panel point, b);

P = an external, concentrated load; P_a, P_b, and P_c, are, respectively, loads suspended from Panel Points a, b, and c;

q_w = total axial tension or compression in a web member;

V = total external shear; V'_{ab} = vertical shear in the upper chord between Panel Points a and b; V'' = vertical shear for the lower chord; the external shear, V, is considered positive when the resultant of the external forces on that part of the truss to the left of the point under consideration, acts upward. Internal moments and forces are considered positive when they act in the direction shown in the illustrations;

w = a subscript denoting "web";

x = a distance measured parallel to the X-axis;

y = a distance measured parallel to the Y-axis;

α = a measure of bending moment ratios $= \dfrac{M'_{ab} + M'_{ba}}{M''_{ab} + M''_{ba}}$;

β = a measure of bending moment ratios $= \dfrac{M'_{ba}}{M''_{ba}}$;

Δ = deflection; Δ_x = horizontal deflection; Δ_y = vertical deflection;

θ = angular deflection; θ_A = angular deflection at Point A;

ϕ = ratio of stiffness, top and bottom chords $= \dfrac{L_i I''_c}{L I'_c}$.

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PAPERS

SIMULTANEOUS EQUATIONS IN MECHANICS SOLVED BY ITERATION

By W. L. Schwalbe,[1] Esq.

Synopsis

Iteration (meaning "repetition") stands for the process of successive approximations applied to the solution of simultaneous equations. A set of simultaneous equations may be characterized by a single equation such as, for instance, the well known equation of three moments for a continuous beam. Such an equation, with given boundary values, is easily solved by elimination or by determinants when the number of unknowns is small, say, eight or less, but requires less unwieldy methods when the number of unknowns is greater than eight. In this paper the method of iteration is applied to the equation of three moments, the equation of three angles, and the equation of five angles. All these equations occur in structural mechanics, the first two in the theory of continuous beams, and the third one in the theory of continuous frames.

The purpose in writing this paper is not to present a new method (all the subject-matter can be found in the references listed in Appendix I); but rather to aid the student in civil engineering in making a co-ordinated progress from the theory of beams in the elementary course in strength of materials to the theory of frames in the more advanced courses in structural civil engineering.

The derivations of formulas introduced are outlined separately in Appendix II and a list of symbols is arranged for convenience of reference in Appendix III. In this notation an effort has been made to conform essentially with "Symbols for Mechanics, Structural Engineering, and Testing Materials"[2] compiled by a committee of the American Standards Association, with Society representation, and approved by the Association in 1932.

Note.—Discussion on this paper will be closed in **October, 1936**, *Proceedings*.

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[2] A. S. A.—*Z10a*—1932.

Equation of Three Moments

The equation of three moments, which relates the statically indeterminate moments at three consecutive supports of a continuous beam (supports fixed and modulus of elasticity of material constant), may be written in the form:

$$\frac{M_{x-1}}{K_x} + 2\,S_x\,M_x + \frac{M_{x+1}}{K_{x+1}} = U_x \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

in which M_x = the bending moment at Support x; K = a stiffness ratio, $\frac{I}{L}$; S_x = a measure of stiffness $= \frac{1}{K_x} + \frac{1}{K_{x+1}}$; and U_x = a load factor in two adjacent spans, such that:

$$U_x = -\frac{6\,Q_{x-1}}{I_x\,L_x} - \frac{6\,Q_{x+1}}{I_{x+1}\,L_{x+1}} \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

in which Q_{x-1} and Q_{x+1} are, respectively, the moment areas below the moment diagram in Spans L_x and L_{x+1} with respect to Supports $x - 1$ and $x + 1$. The span is regarded as simply supported under the given loads.

In order to solve Equation (1) by iteration, or successive approximations, the first approximation is computed for each support, x, as:

$$(M_x)_1 = \frac{U_x}{2\,S_x} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Substituting this value back into Equation (1) the second approximation is found to be:

$$(M_x)_2 = (M_x)_1 - \frac{1}{2\,S_x}\left(\frac{(M_{x-1})_1}{K_x} + \frac{(M_{x+1})_1}{K_{x+1}}\right) \ldots\ldots\ldots\ldots (4)$$

and so on to the nth approximation (n = 1, 2, . . .):

$$(M_x)_n = (M_x)_1 - \frac{1}{2\,S_x}\left[\frac{(M_{x-1})_{n-1}}{K_x} + \frac{(M_{x+1})_{n-1}}{K_{x+1}}\right] \ldots\ldots\ldots\ldots (5)$$

Added speed of convergence is obtained if corrected values are used as soon as they are obtained, for computing the next approximation. Thus, in working from left to right over the series of supports, Equation (5) may be written as:

$$(M_x)_n = (M_x)_1 - \frac{1}{2\,S_x}\left[\frac{(M_{x-1})_n}{K_x} + \frac{(M_{x+1})_{n-1}}{K_{x+1}}\right] \ldots\ldots\ldots\ldots (6)$$

That is, the nth approximation, $(M_{x-1})_n$, is used in computing $(M_x)_n$. The proof of convergence of Equations (5) and (6) to the solution of Equation (1) is given in Appendix II.

According to Equation (6) the first approximation, which is computed from the load factor, U_x, is repeatedly corrected by the factor, $\frac{1}{2\,S_x}\left[\frac{(M_{x-1})_n}{K_x} + \frac{(M_{x+1})_{n-1}}{K_{x+1}}\right]$, until the computed values remain stationary to any required

degree of approximation. The correction factor is easily remembered as the sum of the products of the corrected moments on each side of a support and their corresponding $\frac{1}{K}$ - values in the two adjacent spans, divided by twice the sum of the two $\frac{1}{K}$ - values.

Case 1.—An example of a continuous beam over five supports with end moments zero, is shown in Fig. 1, including the computed values of the

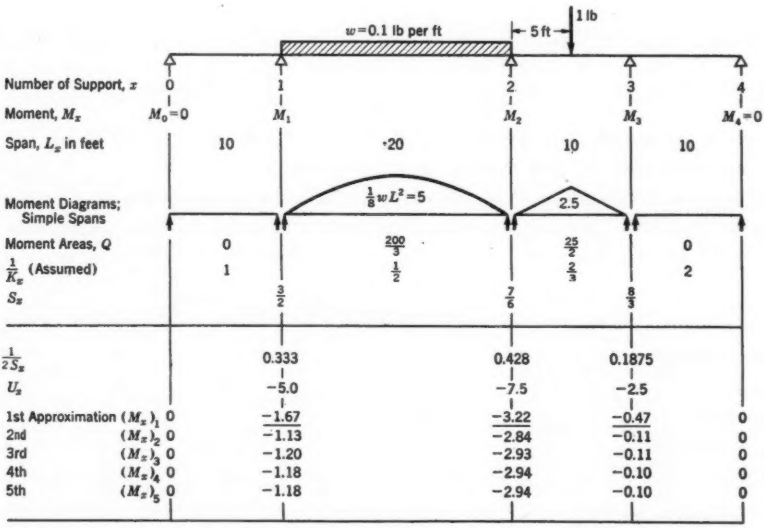

FIG. 1.

different quantities. To illustrate the use of Equation (6), the value, $(M_2)_2 = -2.84$, for $x = 2$, in Fig. 1, is computed as follows: The correction factor to the first approximation, -3.22, is $\left\{(-1.13 \times \frac{1}{2}) + (-0.47 \times \frac{2}{3})\right\} \times 0.428 = -0.38$. Then, $(M_2)_2 = -3.22 - (-0.38) = -2.84$.

As a check, the values of the three moments, computed by elimination from the three simultaneous equations which are found when Equation (1) is applied three times, are: $M_1 = -1.18$; $M_2 = -2.93$; and $M_3 = -0.10$.

EQUATION OF THREE ANGLES

The equation of three angles expresses the relation between the angles of rotation at three successive supports of a continuous beam. The equation is:

$$K_x\,\theta_{x-1} + 2\,(K_x + K_{x+1})\,\theta_x + K_{x+1}\,\theta_{x+1} = \frac{\Delta M'_x}{2\,E} \ldots\ldots (7)$$

in which θ_x = angle of rotation of the beam at Support x (θ is plus (+) when clockwise); $\Delta M'$ = load factor = $(M'_x)_L - (M'_x)_R$; $(M'_x)_L$ = fixed-end

moment at left of support, x, for the span, L_x, with given loads; $(M'_x)_R$ = fixed-end moment at right of support, x, for the span, L_{x+1}, with given loads; and E = modulus of elasticity of material.

The solution of Equation (7) by iteration proceeds in the same manner as that outlined for the equation of three moments. The approximating equation is written as:

$$(\theta_x)_n = (\theta_x)_1 - \frac{[K_x\,(\theta_{x-1})_n + K_{n+1}\,(\theta_{x+1})_{n-1}]}{2\,(K_x + K_{x+1})} \quad \ldots\ldots\ldots\ldots (8)$$

in which $(\theta_x)_n$ is the nth approximation (n = 1, 2, ...), and the first approximation is $(\theta_x)_1 = \dfrac{\Delta M'_x}{2\,E\,(2\,K_x + 2\,K_{x+1})}$. The following approximations are then corrected step by step by means of Equation (8) until the values become stationary to any degree of approximation.

Case 2.—As an illustrative example the same continuous beam as in Case 1 is used (see Fig. 2). From the fundamental equations (see Appendix II,

Number of Support, x	0		1		2		3		4
Rotation, θ_x	$\theta_0 = -\frac{\theta_1}{2}$		θ_1		θ_2		θ_3		$\theta_4 = -\frac{\theta_3}{2}$
K_x (Assumed)		1		2		$\frac{3}{2}$		$\frac{1}{2}$	
Fixed-End Moment, M'_x	0		$0, -\frac{10}{3}$		$-\frac{10}{3}, -\frac{5}{4}$		$-\frac{5}{4}, 0$		0
$\Delta M'_x$	0		$+\frac{10}{3}$		$-\frac{25}{12}$		$-\frac{5}{4}$		0
$\frac{1}{2(K_x+K_{x+1})}$			$\frac{1}{6}$		$\frac{1}{7}$		$\frac{1}{4}$		
1st Approximation $2E(\theta_x)_1$	−0.28		+0.555		−0.2976		−0.3125		+0.15
2nd $2E(\theta_x)_2$	−0.35		+0.70		−0.44		−0.11		+0.05
3rd $2E(\theta_x)_3$	−0.38		+0.76		−0.48		−0.13		+0.065
4th $2E(\theta_x)_4$	−0.39		+0.78		−0.49		−0.13		+0.065
5th $2E(\theta_x)_5$	−0.395		+0.790		−0.494		−0.136		+0.068
6th $2E(\theta_x)_6$	−0.393		+0.786		−0.493		−0.1361		+0.0681

FIG. 2.

Equations (24) and (25)), the boundary values, θ_0 and θ_4, are obtained under the assumption of no restraints at the ends, thus: $\theta_0 = -\dfrac{\theta_1}{2}$; and $\theta_4 = -\dfrac{\theta_3}{2}$.

Computed values are included in Fig. 2. For example, the value, $2\,E\,(\theta_1)_2 = +\,0.70$, is computed as follows: The correction factor is $\frac{1}{6}\left\{(-\,0.28 \times 1) + (-\,0.30 \times 2)\right\} = -\,0.15$; and then, $2\,E\,(\theta_1)_2 = +\,\underline{0.55} - (-\,0.15) = +\,0.70$. Some labor of computation is saved if the corrections are made roughly at first, say, to one decimal place; then, when the values become about stationary, to the second decimal place, etc. From Equations (24) and (25) in Appendix II, the moments at the supports are computed as, $M_1 = -\,1.18$; $M_2 = -\,2.93$; and $M_3 = -\,0.10$.

EQUATION OF FIVE ANGLES; NO SIDE-SWAY

The equation of five angles is an expression of the equilibrium of a rigid joint in a continuous frame, relating the angle of rotation of the joint to the four neighboring angles and the loads on both vertical and horizontal members. For a frame with no side-sway (joints fixed in space), the equation is:

$$K_{x,\,y}\ \theta_{x-1,\,y} + K_{y,\,x}\ \theta_{y-1,\,x} + 2\ \Sigma K\ \theta_{x,\,y} + K_{x+1,\,y}\ \theta_{x+1,\,y} + K_{y+1,\,x}\ \theta_{y+1,\,x} = U_{x,\,y} \qquad (9)$$

in which $\theta_{x,\,y}$ = the angle of rotation at Joint x, y (see Fig. 9(*a*), Appendix II); $K_{x,\,y}$ = stiffness ratio for girder span; $K_{y,\,x}$ = stiffness ratio for column span; $\Sigma K = K_{x,\,y} + K_{x+1,\,y} + K_{y,\,x} + K_{y+1,\,x}$, $U_{x,\,y}$ = load factor $= \dfrac{1}{2\,E}\ (\Delta\, M'_{x,\,y} + \Delta\, M'_{y,\,x})$; $\Delta\, M'_{x,\,y}$ = difference, at Joint x, y, between the fixed-end moments of the girders with vertical loads, $[\Delta\, M'_{x,\,y} = (M'_{x,\,y})_L - (M'_{x,\,y})_R]$; and, $\Delta\, M'_{y,\,x}$ = the difference, at Joint x, y, between the fixed-end moments of the columns with horizontal loads, $[\Delta\, M'_{y,\,x} = (\Delta\, M'_{y,\,x})_L - (\Delta\, M'_{y,\,x})_R]$.

The approximating equation is written as:

$$(\theta_{x,\,y})_n = (\theta_{x,\,y})_1 - \frac{1}{2\ \Sigma K}\ [K_{x,\,y}\ (\theta_{x-1,\,y})_n + K_{x+1,\,y}\ (\theta_{x+1,\,y})_n + K_{y,\,x}\ (\theta_{y-1,\,x})_{n-1} + K_{y+1,\,x}\ (\theta_{y+1,\,x})_{n-1}] \qquad (10)$$

in which $(\theta_{x,\,y})_n$ = the *n*th approximation; and $(\theta_{x,\,y})_1$, the first approximation, $= \dfrac{U_{x,y}}{2\ \Sigma K}$.

The value of θ in Equation (10) that bears the index, *n*, depends on the order in which the correction factors (the quantity in parenthesis) are computed. The order is immaterial of course; any two of the four θ-values may be given the index, *n*. As written, Equation (10) indicates that the order of procedure is from left to right and from top to bottom. It expresses the fact that each value of $\theta_{x,\,y}$ is corrected, successively, by the sum of the products of the four neighboring θ-values and their *K*-factors, divided by twice the sum of the four *K*-values that meet at Point (x, y). The correction is always made from the first approximation, $(\theta_{x,\,y})_1$, which is first computed for each joint from the load factor. The proof of convergence for the procedure is indicated in Appendix II.

Case 3.—Fig. 3 shows a frame, two stories high (3)[3]. The values of the angles at the lower ends of the lower columns are assumed to be zero. For this example, $U_{x,\,y} = \dfrac{\Delta\ M'_{x,y}}{2\ E}$; and $\Delta\ M'_{y,x} = 0$, since there are no horizontal loads. To illustrate the application of Equation (10) the following values

[3] Numerals in parentheses refer to Appendix I.

for $2\,E\,\theta_{o,2}$ and $2\,E\,\theta_{1,2}$ are computed:

$$2\,E\,(\theta_{o,2})_1 = \frac{U_{x,y}}{2\,\Sigma K} = \frac{\Delta\,M'_{o,2}}{2\,(\frac{1}{6}+\frac{1}{3})} = \frac{[0-(-3)]}{1} = +\,\underline{3.00};$$

$$2\,E\,(\theta_{o,2})_2 = +\,\underline{3.00} - \frac{1}{1}\left\{(-\,1.80\times\tfrac{1}{3}) + (0.00\times\tfrac{1}{6})\right\} = +\,3.60;$$

and,

$$(\theta_{1,2})_2 = -\,\underline{1.80} - \frac{1}{\frac{5}{3}}\left\{(+\,3.60\times\tfrac{1}{3}) + (\underline{1.50}\times\tfrac{1}{6}) + (\tfrac{1}{3}\times\underline{1.80})\right\} = -\,3.03$$

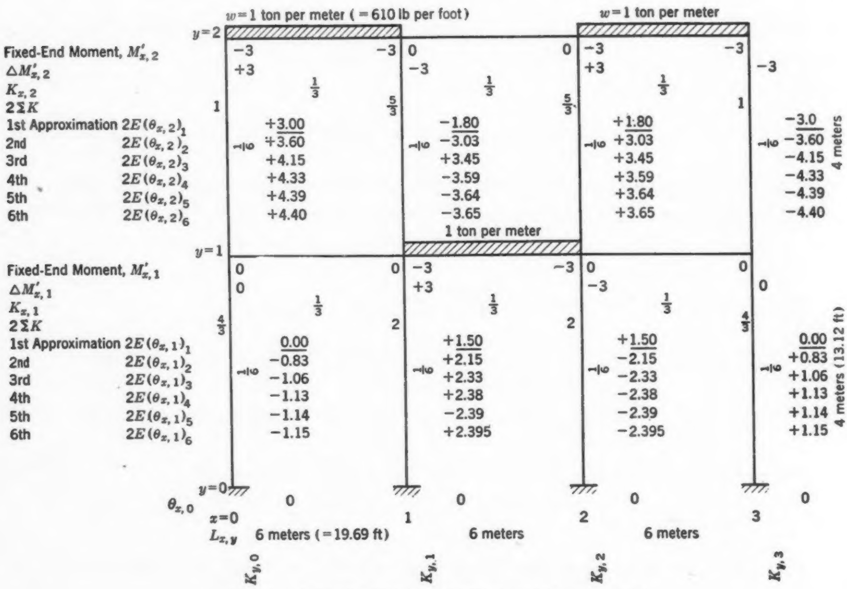

FIG. 3.—SUCCESSIVE APPROXIMATIONS FOR $2E\theta$

The procedure is continued until the values become stationary. On account of symmetry in this example computations for only one-half the joints need be made. The results shown in Fig. 3 check the values determined by elimination.

EQUATION OF FIVE ANGLES—SIDE-SWAY

For a continuous frame acted upon by horizontal loads, the rigid joints may undergo a horizontal displacement, or side-sway. The equation of five angles for this case is:

$$K\,(\theta) = U_{x,y} \qquad (11)$$

in which,

$$K\,(\theta) = K_{x,y}\,\theta_{x-1,y} + K_{y,x}\,\theta_{y-1,x} + 2(\Sigma K)\,\theta_{x,y} + K_{x+1,y}\,\theta_{x+1,y} + K_{y+1,x}\,\theta_{y+1,x} \qquad (12)$$

and,

$$U_{x,\,y} = \frac{\Delta\, M'_{y,x}}{2\,E} + 3\ K_{y,\,x}\ (\alpha_{y,\,x})_L + 3\ K_{y+1,\,x}\ (\alpha_{y+1,\,x})_L \ldots\ldots (13)$$

ΣK = sum of four K-values of members meeting at a joint; and $\Delta\, M'_{y,\,x}$ = difference in fixed-end moments for the column ends meeting at Joint x, y; and $(\alpha_{y,\,x})_L$ = angle of column rotation due to the horizontal displacement of the upper end, y, relative to the lower end, $y - 1$, of the column, $L_{y,\,x}$. The subscript, L, indicates that the angle is to the left of a joint when the observer is to the right of the column (Fig. 10(a), Appendix II).

For convenience in computing Angles θ and α, the former is written $\theta = \theta_b + \theta_s$. If θ_b represents the angle of rotation due to the bending action alone of the horizontal loads on the columns (side-sway being zero), and θ_s is the rotation due to the side-sway alone, Equation (11) may be replaced by:

$$K\ (\theta_b) = \frac{\Delta\ M'_{y,x}}{2\ E} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14)$$

and,

$$K\ (\theta_s) = 3\ K_{y,\,x}\ (\alpha_{y,\,x})_L + 3\ K_{y+1,\,x}\ (\alpha_{y+1,\,x})_L \ldots\ldots\ldots (15)$$

Since the angles, α, as well as the angles, θ_s, in Equation (15) are unknown, it is necessary to write an additional equation of equilibrium for the horizontal forces above each floor level; thus:

$$\sum_{y-1}^{y=k}\sum_{x=0}^{x=m} P_{y,x} - \sum_{x=0}^{x=m} (R_{y-1,x})_{o,R} + \sum_{x=0}^{x=m} \frac{(M'_{y-1,x})_R - (M'_{y,x})_L}{L_{y,x}}$$

$$= - \sum_{x=0}^{x=m} \frac{6\ E\ K_{y,x}}{L_{y,x}}\ [\theta_{x,y} + \theta_{y-1,x} - 2\ (\alpha_{y,x})_L] \ldots\ldots\ldots\ldots (16)$$

in which k = number of floor levels minus one; $P_{y,\,x}$ = horizontal load on the column, $L_{y,\,x}$; m = number of columns minus one; $(R_{y-1,\,x})_{o,R}$ = horizontal reaction at joint on the level, $y - 1$, due to given loads on the columns, with end moments equal to zero; and M' = fixed-end moment, either to the right or to the left of a joint, as indicated by the index, R, or L (Fig. 11, Appendix II).

Case 4.—As an example of the application of Equations (14), (15), and (16) to a frame with side-sway, the frame in Case 3 is subjected to horizontal loads, w, equal to 2 tons per m (1 220 lb per ft; see Fig. 4). The first part of the problem is the solution of Equation (14). This is accomplished as outlined in Case 3 for Equation (9). The fixed-end moments, $M'_{y,\,x}$, are $-\frac{1}{12}\ w\ L^2 = -\frac{8}{3}$ and the first approximation is, $2\ E\ (\theta_b)_1 = \frac{\Delta\,M'_{y,x}}{2\ \Sigma\ K}$. In Fig. 4 the first approximations are underlined and the succeeding ones are tabulated below the first.

For the solution of Equation (15), subject to the condition expressed by Equation (16), it is assumed that the angle of side-sway, α, is the same for

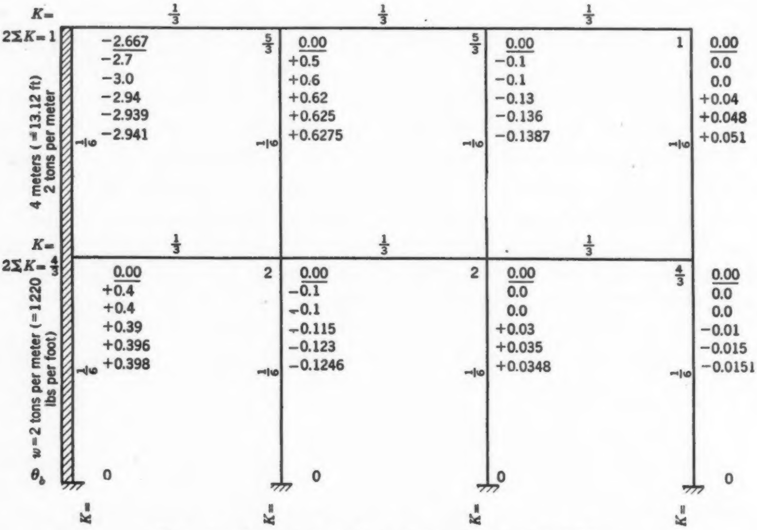

FIG. 4.—SUCCESSIVE APPROXIMATIONS FOR $2E\theta_b$

each column in any given story. Since, for this frame, $K_{y,\,x} = K_{y+1,\,x} = \frac{1}{6}$ for all columns, Equation (15) may be written:

$$K\,(\theta_s) = \frac{1}{2}\,(\alpha_{y\text{-}1})_L + \frac{1}{2}\,(\alpha_y)_L \ldots\ldots\ldots\ldots\ldots\ldots(17)$$

The joint rotation, θ_s, may be further divided into: $(\theta_s)_{y\text{-}1}$, the joint rotation corresponding to a side-sway $(\alpha_{y\text{-}1})_L$; and $(\theta_s)_y$, the joint rotation corresponding to a side-sway, $(\alpha_y)_L$. Consequently, for $y = 1, 2$ and $(\alpha_0)_L = 0$, Equation (17) may be replaced by:

$$K\,(\theta_s)_{y\text{-}1} = \frac{1}{2}\,(\alpha_1)_L = \frac{1}{2}\,\alpha_1 \ldots\ldots\ldots\ldots\ldots\ldots(18)$$

and,

$$K\,(\theta_s)_y = \frac{1}{2}\,(\alpha_1)_L + \frac{1}{2}\,(\alpha_2)_L = \frac{1}{2}\,\alpha_1 + \frac{1}{2}\,\alpha_2 \ldots\ldots\ldots\ldots(19)$$

Equations (18) and (19) are solved by iteration, as outlined in Case 3, first for the values, $\alpha_1 = \alpha_2 = \dfrac{1.00}{E}$. The results are shown in Figs. 5 and 6. These values, for $\alpha = 1$, are then used in solving Equation (16), which, for Case 4, reduces to:

$$\alpha_1 = \frac{1}{8}\,(\theta_{0,\,1} + \theta_{1,\,1} + \theta_{2,\,1} + \theta_{3,\,1}) + \frac{6}{E} \ldots\ldots\ldots\ldots(20)$$

at the first story; and,

$$\alpha_2 = \frac{1}{8}\,(\theta_{0,\,1} + \theta_{1,\,1} + \theta_{2,\,1} + \theta_{3,\,1} + \theta_{0,\,2} + \theta_{1,\,2} + \theta_{2,\,2} + \theta_{3,\,2}) + \frac{2}{E}\,.\,(21)$$

at the second story.

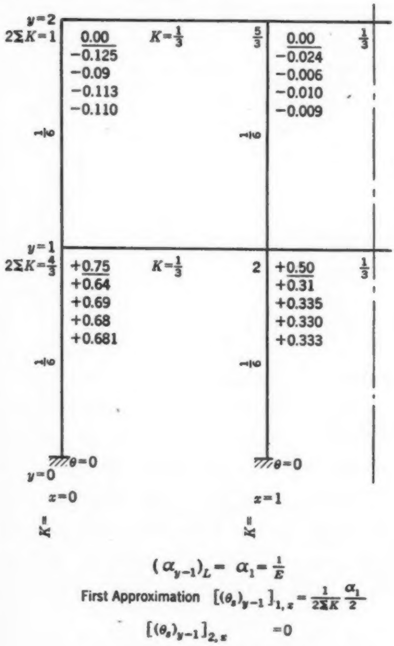

FIG. 5.—VALUES OF $2E(\theta_s)_{y-1}$

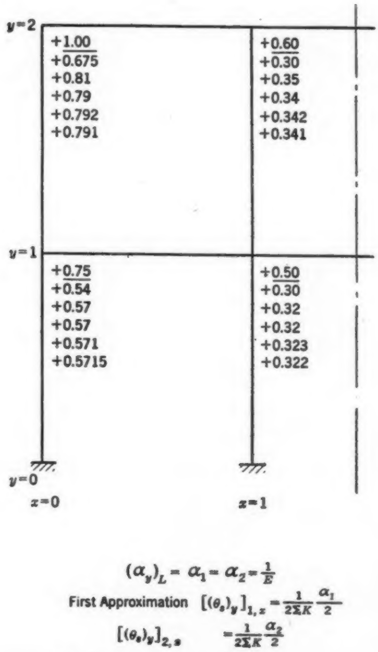

FIG. 6.—VALUES OF $2E(\theta_s)_y$

In solving Equations (20) and (21) the first approximation for the α-values is obtained by placing all the θ-values equal to zero or $(\alpha_1)_1 = \dfrac{6}{E}$ and $(\alpha_2)_1 = \dfrac{2}{E}$. These values are then corrected by multiplying the values of $(\theta_s)_{y-1}$ and $(\theta_s)_y$ for unit rotation, α, by the first approximation for α and using these new values of $(\theta_s)_{y-1}$ and $(\theta_s)_y$, together with the θ_b-values (see Fig. 4), for a second approximation of α according to Equations (20) and (21), etc. To illustrate, the second approximations are:

$$(\alpha_1)_2 = \frac{1}{8}\left[(0.681 + 0.333)\,\frac{6}{E} + (0.571 + 0.322)\,\frac{2}{E} + (0.398 - 0.125 + 0.035 - 0.015)\,\frac{1}{2\,E}\right] + \frac{6}{E} = \frac{7.002}{E};$$ and, $$(\alpha_2)_2 = \frac{1}{8}\left[(0.681 + 0.333 - 0.110 - 0.009)\,\frac{7.002}{E} + (0.571 + 0.322 + 0.791 + 0.341)\,\frac{2}{E} + (0.398 - 0.125\right.$$

$$+ 0.035 - 0.015 - 2.941 + 0.628 - 0.139 + 0.051) \frac{1}{2\,E}\Bigg] + \frac{2}{E} = \frac{3.16}{E}.$$

Successive values of α up to the sixth approximation are shown in Table 1.

TABLE 1.—Successive Approximations for α (Rotation Due to Side-Sway)

Approximation No.	$y = 1$		$y = 2$	
	Symbols	Value	Symbols	Value
1	$E\ (\alpha_1)_1$	6.00	$E\ (\alpha_2)_1$	2.00
2	$E\ (\alpha_1)_2$	7.00	$E\ (\alpha_2)_2$	3.16
3	$E\ (\alpha_1)_3$	7.26	$E\ (\alpha_2)_3$	3.48
4	$E\ (\alpha_1)_4$	7.33	$E\ (\alpha_2)_4$	3.57
5	$E\ (\alpha_1)_5$	7.35	$E\ (\alpha_2)_5$	3.60
6	$E\ (\alpha_1)_6$	7.353	$E\ (\alpha_2)_6$	3.601

APPENDIX I

List of References

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APPENDIX II

Mathematical Derivations

The fundamental equations relating moments and slopes for a single simply supported span (Fig. 7) are:

$$\theta_A = (\theta_A)_0 + \frac{M_A L}{3 E I} + \frac{M_B L}{6 E I} \quad \text{.......... (22)}$$

and,

$$\theta_B = (-\theta_B)_0 - \frac{M_B L}{3 E I} - \frac{M_A L}{6 E I} \quad \text{.......... (23)}$$

or,

$$M_A = 2\, E\, K\, (2\, \theta_A + \theta_B) + M'_A \quad \text{.......... (24)}$$

and,

$$M_B = -\, 2\, E\, K\, (2\, \theta_B + \theta_A) + M'_B \quad \text{.......... (25)}$$

in which the subscript, 0, denotes that the moments at the ends are zero. Rotations are positive when clockwise; and moments are positive as shown in Fig. 7.

Equation of Three Moments.—The equation for three moments is found by applying Equations (22) and (23) to two adjacent spans in a continuous beam and making $(\theta_x)_L = (\theta_x)_R = \theta_x$ (see Fig. 8), in which $(\theta_x)_L$ is the angle of rotation at the left of Support x, and $(\theta_x)_R$ is the angle of rotation at the right of Support x. In a continuous beam these two rotations are equal. Expressing the angles, θ, in terms of the area-moments, Q_{x-1}, and Q_{x+1}, the equation of three moments becomes (7) (see Equation (1)):

$$\frac{M_{x-1}}{K_x} + 2\, M_x \left(\frac{1}{K_x} + \frac{1}{K_{x+1}}\right) \frac{M_{x+1}}{K_{x+1}} = -\frac{6\, Q_{x-1}}{I_x\, L_x} - \frac{6\, Q_{x+1}}{I_{x+1}\, L_{x+1}} \quad \text{.... (26)}$$

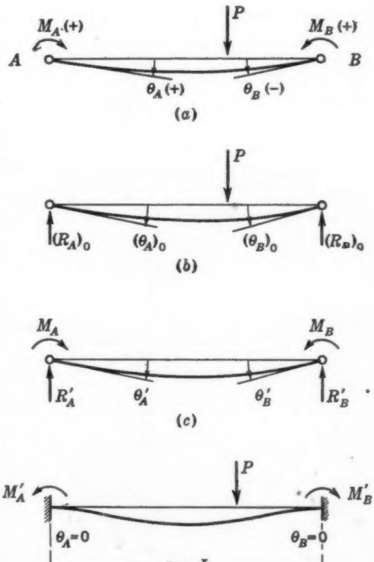

FIG. 7.

Equation of Three Angles.—The equation of three angles is obtained in a manner similar to Equation (26) by applying Equations (24) and (25) to two adjacent spans (Fig. 8) and equating the moments on either side of Support x; that is,

$$(M_x)_L = (M_x)_R = M_x \quad \text{.......... (27)}$$

in which $(M_x)_L$ is the moment at x in Member L_x to the left of x and $(M_x)_R$ is the moment at x in Member L_{x+1} to the right of x. Then (3) (see Equation (7));

$$K_x\ \theta_{x-1} + 2\ (K_x + K_{x-1}) + K_{x+1}\ \theta_{x+1} = \frac{\Delta\ M'_x}{2\ E} \quad \ldots\ldots\ldots(28)$$

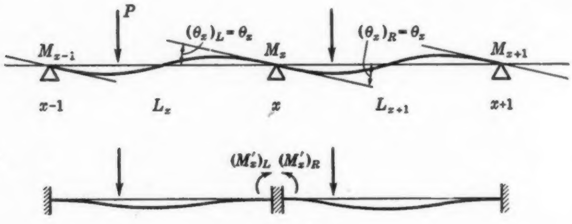

FIG. 8

in which $\Delta M'_x = (M'_x)_L - (M'_x)_R =$ the difference in fixed-end moments at x (see Fig. 8).

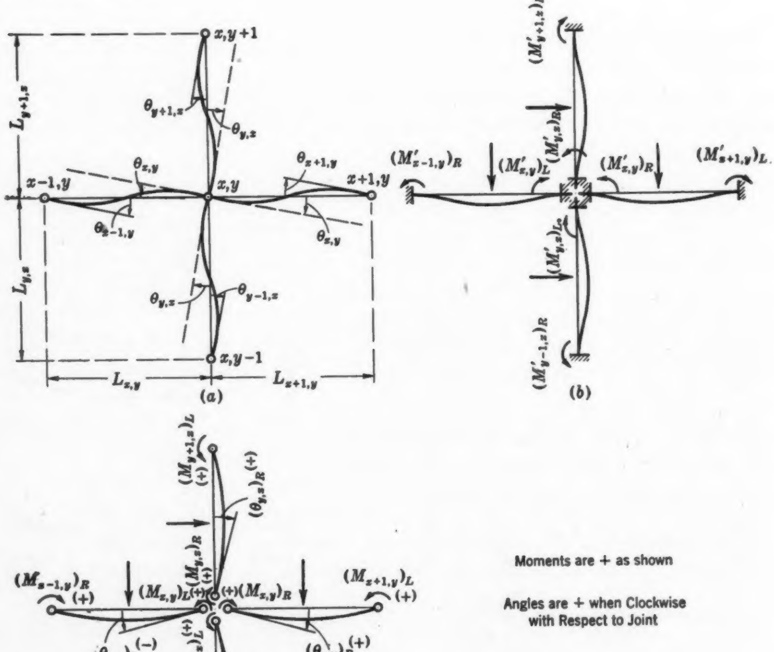

FIG. 9.

Equation of Five Angles: No Side-Sway.—Fig. (9*a*) shows Joint (x, y) rotated through a positive angle, $\theta_{x,y}$. Fig. 9(*b*) shows the fixed-end moments under the given loads. If Equations (24) and (25) are applied to the four spans joining at (x, y) (Fig. 9(*c*)), the following equations are obtained:

In Span $L_{x,y}$:

$$(M_{x,y})_L = -2\,E\,K_{x,y}\,[2\,(\theta_{x,y})_L + (\theta_{x+1,y})_R] + (M'_{x,y})_L \ldots (29)$$

In Span $L_{x+1,y}$:

$$(M_{x,y})_R = 2\,E\,K_{x+1,y}\,[2\,(\theta_{x,y})_R + (\theta_{x+1,y})_L] + (M'_{x,y})_R \ldots (30)$$

In Span $L_{y,x}$:

$$(M_{y,x})_L = -2\,E\,K_{y,x}\,[2\,(\theta_{y,x})_L + (\theta_{y-1,x})_R] + (M'_{y,x})_L \ldots (31)$$

and, in Span $L_{y+1,x}$:

$$(M_{y,x})_R = 2\,E\,K_{y+1,x}\,[2\,(\theta_{y,x})_R + (\theta_{y+1,x})_L] + (M'_{y,x})_R \ldots (32)$$

Substituting Equations (29) to (32) in the equilibrium equation,

$$(M_{x,y})_R + (M_{y,x})_R = (M_{x,y})_L + (M_{y,x})_L \ldots\ldots\ldots\ldots (33)$$

and, making use of the equality of angles at Joint x, y,

$$(\theta_{x,y})_R = (\theta_{x,y})_L = (\theta_{y,x})_L = (\theta_{y,x})_R = \theta_{x,y} \ldots\ldots\ldots (34)$$

the five-angle equation (3) (see Equation (9)), is obtained; thus:

$$K_{x,y}\,\theta_{x-1,y} + K_{y,x}\,\theta_{y-1,x} + 2\,\theta_{x,y}\,(K_{x,y} + K_{x+1,y} + K_{y,x} + K_{y+1,x})$$
$$+ K_{x+1,y}\,\theta_{x+1,y} + K_{y+1,x}\,\theta_{y+1,x} = \frac{1}{2\,E}\,[\Delta\,M'_{x,y} + \Delta\,M'_{y,x}] \ldots\ldots (35)$$

in which (see Fig. 9(*b*)): $\Delta\,M'_{x,y} = (M'_{x,y})_L - (M'_{x,y})_R$; and $\Delta\,M'_{y,x} = (M'_{y,x})_L - (M'_{y,x})_R$.

Equation of Five Angles: With Side-Sway.—It is assumed that no vertical loads act on the girders. Fig. 10(*a*) shows the horizontal displacement of Joints $x, y + 1$ and $x, y - 1$, relative to Joint x, y. The angles of displacement are $(\alpha_{y,x})_R$ to the right of Joint x, y, and $(\alpha_{y,x})_L$ to the left of Joint x, y. The other angles involved are shown in Fig. 10. Applying Equations (24) and (25) to the members in Fig. 10(*c*), the following equations are obtained:

In Span $L_{x,y}$:

$$(M_{x,y})_L = -2\,E\,K_{x,y}\,[2\,(\theta_{x,y})_L + (\theta_{x-1,y})_R] \ldots\ldots\ldots (36)$$

In Span $L_{x+1,\, y}$:

$$(M_{x,\, y})_R = 2\, E\, K_{x+1,\, y}\, [2\, (\theta_{x,\, y})_R + (\theta_{x+1, y})_L] \quad \ldots\ldots\ldots (37)$$

In Span $L_{y,\, x}$:

$$(M_{y,\, x})_L = -\, 2\, E\, K_{y,\, x}\, [2\, (\omega_{y,\, x})_L + (\omega_{y-1, x})_R] + (M'_{y,\, x})_L \quad \ldots (38)$$

and, in Span $L_{y+1,\, x}$:

$$(M_{y,\, x})_R = 2\, E\, K_{y+1, x}\, [\, 2\, (\omega_{y,\, x})_R + (\omega_{y+1,\, x})_L] + M'_{y,\, x})_R \quad \ldots (39)$$

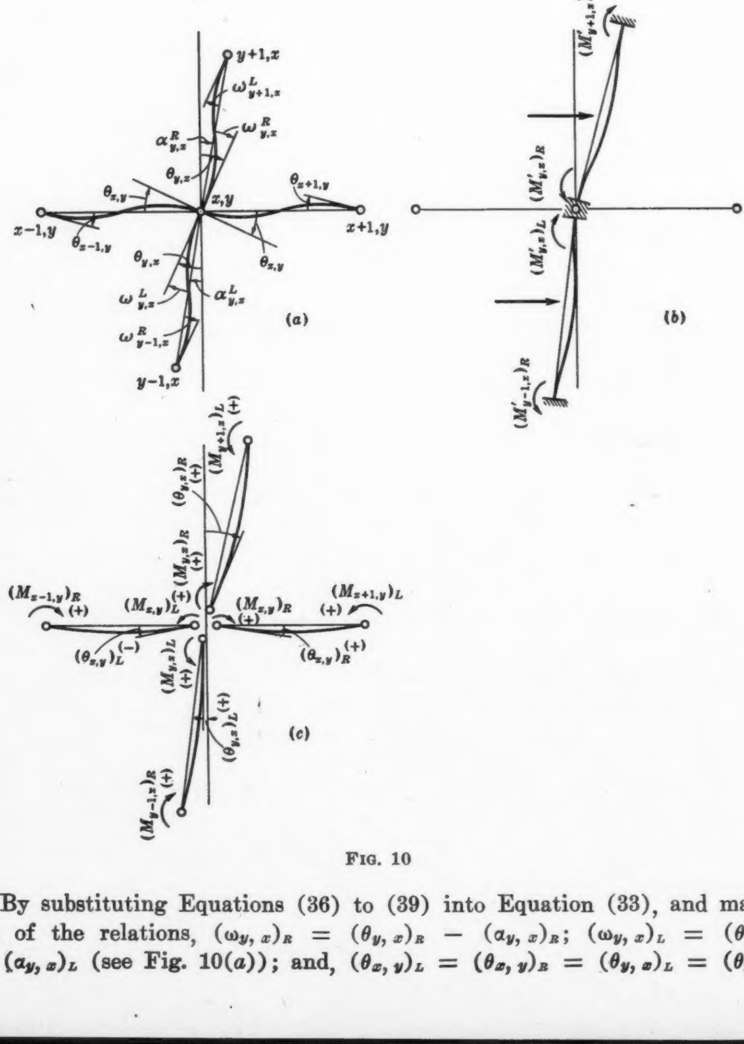

FIG. 10

By substituting Equations (36) to (39) into Equation (33), and making use of the relations, $(\omega_{y,\, x})_R = (\theta_{y,\, x})_R - (\alpha_{y,\, x})_R$; $(\omega_{y,\, x})_L = (\theta_{y,\, x})_L - (\alpha_{y,\, x})_L$ (see Fig. 10(*a*)); and, $(\theta_{x,\, y})_L = (\theta_{x,\, y})_R = (\theta_{y,\, x})_L = (\theta_{y,\, x})_R$

$= \theta_{x,y}$, the equation of five angles, for horizontal loads, with side-sway, is obtained; thus (see Equation (11)):

$$K_{x,y}\,\theta_{x-1,y} + K_{x+1,y}\,\theta_{x+1,y} + 2\,\theta_{x,y}\,(K_{x,y} + K_{y,x} + K_{x+1,y} + K_{y+1,x})$$

$$+ K_{y,x}\,\theta_{y-1,x} + K_{y+1,x}\,\theta_{y+1,x} = \frac{\Delta M'_{y,x}}{2E} + 3\,K_{y,x}\,(\alpha_{y,x})_L + 3\,K_{y+1,x}\,(\alpha_{y+1,x})_L \text{.} \quad (40)$$

Equation (16) expressing the equilibrium of horizontal forces for the part of the frame above a horizontal section through any floor level is derived as follows: The fundamental relations between moments and reactions (see Fig. 7) are:

$$R_A = (R_A)_o + \frac{M_B - M_A}{L} \dots\dots\dots\dots\dots\dots (41)$$

and,

$$R_B = (R_B)_o + \frac{M_A - M_B}{L} \dots\dots\dots\dots\dots\dots (42)$$

If Equation (41) is applied to a single column (see Fig. 11(a)):

$$(R_{y-1,x})_R = (R_{y-1,x})_{o,R} + \frac{(M_{y,x})_L - (M_{y-1,x})_R}{L_{y,x}} \dots\dots\dots (43)$$

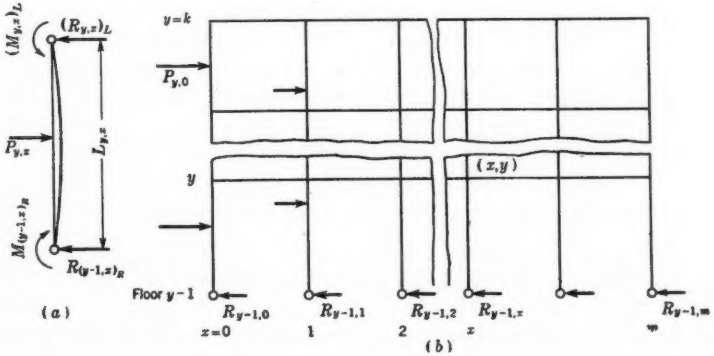

FIG. 11.

From Equations (38) and (39):

$$(M_{y,x})_L - (M_{y-1,x})_R = -6\,E\,K_{y,x}\,[\theta_{x,y} + \theta_{y-1,x} - 2\,(\alpha_{y,x})_L]$$

$$- [(M'_{y-1,x})_R - (M'_{y,x})_L] \dots\dots\dots\dots\dots\dots (44)$$

Since the horizontal loads and reactions are in equilibrium (see Fig. 11(b)):

$$\sum_{x=0}^{x=m} \sum_{y-1}^{y=k} P_{y,x} = \sum_{x=0}^{x=m} (R_{y-1,x})_R \dots\dots\dots\dots\dots (45)$$

Finally, substituting Equations (43) and (44) into Equation (45) (see Equation (16)):

$$\sum_{x=0}^{x=m}\sum_{y-1}^{y=k} P_{y,x} = \sum_{x=0}^{x=m} (R_{y-1,x})_{o,R} - \sum_{x=0}^{x=m} \frac{6\,E\,K_{y,x}}{L_{y,x}} \left[\theta_{x,y} + \theta_{y-1,x} - 2\,(\alpha_{y,x})_L\right]$$

$$- \sum_{x=0}^{x=m} \frac{(M'_{y-1,x})_R - (M'_{y,x})_L}{L_{y,x}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (46)$$

Convergence Proof.—Equation of Three Moments.—That the sequence, $(M_x)_n$ (Equation (5)), is convergent for increasing integral values of n is demonstrated if it can be shown that the sequence of differences between successive approximations converges to zero. It is assumed that the coefficients, $\frac{1}{K}$, are positive and that $S_x = \frac{1}{K_x} + \frac{1}{K_{x+1}}$ does not equal zero; in other words, the $\frac{1}{K}$ - values are not zero for any two adjacent spans in a continuous beam. The nth approximation is expressed by Equation (5) of the paper and the $(n - 1)$th is:

$$(M_x)_{n-1} = (M_x)_1 - \frac{1}{2\,S_x}\left[\frac{(M_{x-1})_{n-2}}{K_x} + \frac{(M_{x+1})_{n-2}}{K_{x+1}}\right] \quad \ldots\ldots\ldots (47)$$

The difference between Equations (5) and (47) is:

$$(M_x)_n - (M_x)_{n-1} = - \frac{1}{2\,K_x\,S_x}\left[(M_{x-1})_{n-1} - (M_{x-1})_{n-2}\right]$$

$$- \frac{1}{2\,K_{x+1}\,S_x}\left[(M_{x+1})_{n-1} - (M_{x+1})_{n-2}\right] \ldots\ldots\ldots\ldots\ldots\ldots (48)$$

For $n = 2$, since $(M_{x-1})_0 = (M_{x+1})_0 = 0$, the difference is:

$$(M_x)_2 - (M_x)_1 = \frac{1}{2\,K_x S_x}(M_{x-1})_1 - \frac{1}{2\,K_{x+1}\,S_x}(M_{x+1})_1 \ldots\ldots\ldots (49)$$

Let M be the largest absolute value among all the moments in the first approximations, including boundary values. Then, the difference, $(M_x)_2 - (M_x)_1$, lies between $+\ 0.5\ M$ and $-\ 0.5\ M$. Writing the difference for $n = 3$:

$$(M_x)_3 - (M_x)_2 = - \frac{1}{2\,S_x\,K_x}\left[(M_{x-1})_2 - (M_{x-1})_1\right]$$

$$- \frac{1}{2\,S_x K_{x+1}}\left[(M_{x+1})_2 - (M_{x+1})_1\right] \ldots\ldots\ldots\ldots\ldots\ldots (50)$$

it is seen that this difference lies between $+\ 0.25\ M$ and $-\ 0.25\ M$. In general, the difference expressed by Equation (48) lies between $\frac{+\,M}{2^{n-1}}$ and $\frac{-\,M}{2^{n-1}}$.

Hence, this difference approaches zero for increasing values of n. To show that $(M_x)_n$ approaches the solution, M_x, expressed by Equation (1), the difference is written in the form $\left(\text{remembering that } (M_x)_1 = \dfrac{U_x}{2\,S_x}\right)$:

$$(M_x)_n - (M_x)_{n-1} = -\frac{1}{2\,S_x}\left[\frac{(M_{x-1})_{n-1}}{K_x} + 2\,(M_x)_{n-1}\,S_x + \frac{(M_{x+1})_{n-1}}{K_{x+1}} - U_x\right] \text{.(51)}$$

Since $(M_x)_n - (M_x)_{n-1}$ approaches zero, the right-hand expression approaches zero; hence, $(M_x)_{n-1}$ satisfies the original Equation (1) for the condition that n becomes infinite.

Convergence is easily shown also for Equation (6) when $(M_{x-1})_n$ is used in computation instead of $(M_{x-1})_{n-1}$; thus:

$$(M_x)_n - (M_x)_{n-1} = -\frac{1}{2\,K_x\,S_x}\left[(M_{x-1})_n - (M_{x-1})_{n-1}\right]$$

$$-\frac{1}{2\,K_{x+1}\,S_x}\left[(M_{x+1})_{n-1} - (M_{x+1})_{n-2}\right] \ldots\ldots\ldots\ldots\ldots(52)$$

and, for $n = 2$:

$$(M_x)_2 - (M_x)_1 = -\frac{1}{2\;K_x\,S_x}\left[(M_{x-1})_2 - M_{x-1})_1\right] - \frac{1}{2\;K_{x+1}\,S_x}(M_{x+1})_1 \text{.(53)}$$

By making use of the upper and lower bounds, as stated for Equation (48) (that is, $(M_x)_2 - (M_x)_1$ is less than $+\,0.5\,M$ and greater than $-\,0.5\,M$), it is also true that $(M_{x-1})_2 - (M_{x-1})_1$ is less than $+\,M$ and greater than $-\,M$. Hence, for Equation (53), $(M_{x-1})_2 - (M_{x-1})_1$ is less than $+\,0.5\,M$ and greater than $-\,0.5\,M$; and, in general, $(M_x)_n - (M_x)_{n-1}$ is less than $\dfrac{+\,M}{2^{n-1}}$ and greater than $\dfrac{-\,M}{2^{n-1}}$, as before.

For the equation of five angles the proof of convergence is easily obtained by generalizing the proof as given for the equation of three moments; in the equation of five angles the value, $\Sigma\,K$, must not be equal to, or less than, zero. A proof of convergence based upon the minimum of a positive definite quadratic form associated with a set of simultaneous equations is given in the paper (4).

APPENDIX III

NOTATION

The following letter symbols, introduced in the paper, are arranged herein, for convenience of reference and for the guidance of discussers:

b = a subscript denoting "bending".

I = rectangular moment of inertia; I_x = moment of inertia in Span L_x, etc.

K = a stiffness ratio, $\frac{I}{L}$; ΣK = a sum of four stiffness ratios $= K_{x,y} + K_{x+1,y} + K_{y,x} + K_{y+1,x}$ at Joint (x, y).

k = number of floor-levels minus one.

L = length; L_x = span length between Supports $x - 1$ and x, etc.; as a subscript, L, denotes left and as a subscript to α, it indicates that the angle is to the left of a joint when the observer is to the right of a column.

M = moment of force; M_x = bending moment at Support x, etc.; M' = fixed-end moment; $\Delta M'_x$ = a local factor = a difference, $(M'_x)_L - (M'_x)_R$, between fixed-end moments at the left and right of Support x.

m = number of columns minus one.

n = a limiting number.

P = a horizontal force applied at any story; P_{yx} = horizontal force between floor levels, $y - 1$ and y.

Q = statical moment of area about a given axis; Q_{x-1} and Q_{x+1} are, respectively, the moment areas below the moment diagram in Spans L_x and L_{x+1} with respect to Supports $x - 1$ and $x + 1$.

R = a subscript denoting "right side".

S = a sum of two stiffness ratios; $S_x = \frac{1}{K_x} + \frac{1}{K_{x+1}}$.

s = a subscript denoting "side-sway".

U = a load factor in two adjacent spans.

α = angle of rotation of one end of one member; $(\alpha_{y,x})_L$ = angle of column rotation due to the horizontal displacement of the upper end, y, with respect to the lower end, $y - 1$, of the column, $L_{y,x}$; the subscript, L, applied to α indicates that the angle is to the left of a joint when the observer is to the right of a column.

Δ = a difference; $\Delta M'_x$ = a load factor = difference, $(M'_x)_L - (M'_x)_R$, between fixed-end moments.

θ = rotation of a joint; θ_b = the rotation due entirely to the bending action of horizontal loads on the columns; θ_s = the rotation due entirely to side-sway.

ω = an angle $= \theta - \alpha$.

AMERICAN SOCIETY OF CIVIL ENGINEERS

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DISCUSSIONS

TAPERED STRUCTURAL MEMBERS: AN ANALYTICAL TREATMENT

Discussion

BY MESSRS. A. W. FISCHER, AND L. LEGENS

A. W. FISCHER,[59] ESQ. (by letter).[59a]—A very thorough mathematical treatment for solving the values of M_A, C_{AB}, and S_A, is presented in this paper. The results are reliable when the substitute I-curve practically coincides with the actual I-curve. By solving for the shape exponent, n, at several points along the tapered member a value can be selected such that the substitute I-curve will fit the actual I-curve very well. Members of uniform depth, with straight-line haunches at either end, or at both ends, should be divided into sections so as to fit the substitute I-curve more closely to the actual I-curve. The taper modulus, A, can be determined readily, but the accurate determination of n is another problem. Since an average can be selected, however, so that the substitute I-curve will give reliable results, it seems that the authors have added a very important item in the theory pertaining to this subject.

For the solution of general tapered members used in ordinary practice it seems that the values of the constants, M_A, C_{AB}, and S_A, can be taken directly from charts[60] in just a fraction of the time required to calculate the constants using the authors' method; and especially is this true for straight haunched beams in which the haunch does not extend to the center of the span.

As an example, consider the symmetrical rigid-frame concrete bridge[61], shown in Fig. 28. If a width of 12 ft is used, $I = d^3$, in feet. Substituting in Equation (7): $A = \dfrac{76.77 - 5.359}{5.359} = 13.33$ = taper modulus for the horizontal member.

NOTE.—The paper by Walter H. Weiskopf and John W. Pickworth, Assoc. Members, Am. Soc. C. E., was published in October, 1935, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: February, 1936, by Messrs. Fred L. Plummer, and LeRoy W. Clark; March, 1936, by Messrs. E. G. Paulet, J. Charles Rathbun, and Halvard W. Birkeland; and May, 1936, by Messrs. C. W. Dunham, Fang-Yin Tsai, A. A. Eremin, and Austin H. Reeves.

[59] Care, Pennsylvania Sugar Co., Philadelphia, Pa.

[59a] Received by the Secretary May 11, 1936.

[60] See "Design of Continuous Frames Having Variable Moments of Inertia", *Civil Engineering*, October, 1932, pp. 647–648.

[61] "Analysis of Rigid Frame Concrete Bridges", Portland Cement Assoc., Third Edition, 1935.

At a point 12 ft from the left support for the horizontal member the depth = 2.65 ft, from which I_e = 18.61, and substituting in Equation (8), $n = \log \frac{76.77 - 18.61}{13.33 \times 18.61} \div \log \frac{12}{30} = 1.583$ = the shape exponent for the horizontal member. In solving the value of n at other points, it seems that the

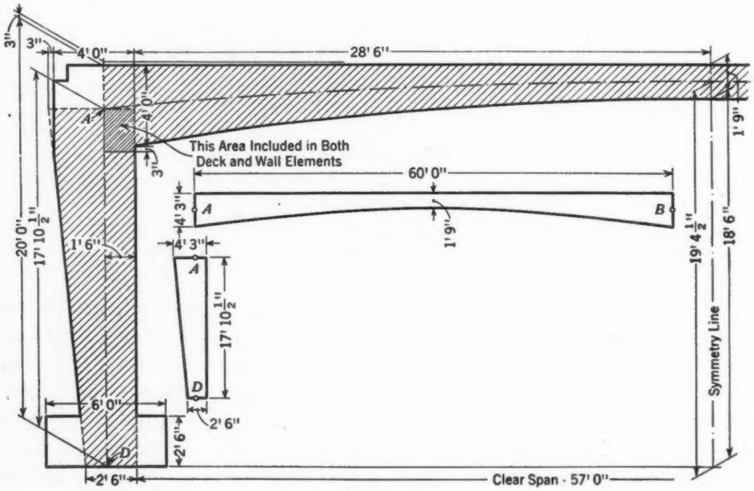

FIG. 28.

value, 1.583, is somewhat too great, and a value of 1.55 is selected so that the substitute I-curve will fit quite well with the actual I-curve for the horizontal member.

Case 2 will apply for the horizontal member and, from Equation (72), F_1 = 1.3117; from Equation (73), with n = 1.55, F_2 = 1.8023; and, from Equation (74), $F_3 = F_2$.

For a uniformly distributed load of w = 90 lb per lin ft, on a span of l = 60 ft, with n = 1.55, Equation (82) yields F_8 = 0.32797, and, from Equation (81), $M_A = M_B = -34\,120$ ft-lb.

For a concentrated load of 2 500 lb at the center of the span, Equation (80) gives F_{10} = 0.5319, and, from Equation (79), with P = 2 500 lb. $M_A = M_B = -25\,620$ ft-lb. Therefore, the total, live load, fixed-end moment $= -34\,120 - 25\,620 = -59\,740$ ft-lb. From Equation (52), $C_{AB} = -\frac{F_1}{F_2} = -0.7278$; and, from Equation (61), with I_A = 76.77, S_A = 0.3775.

For the vertical member, from Equation (7), A = 3.912. At a point 6 ft from the top of the member the thickness = 3.66 ft, and I_e = 49.03. Substituting in Equation (8), with e = 6 and r = 17.88, n = 1.771. In solving for the value of n at other points, it seems that 1.771 is slightly too small; therefore, a value of 1.8 will be used. The vertical member comes under Case 3, and, from Equation (85), F_2 = 0.4865.

Variation of the Cross Method When One End Is Hinged.—From Equation (110), $S'_A = 2.206$. As the horizontal member is symmetrical and $M_A = M_B$, a modified S_A can be used[62] so that one distribution is all that is required. Calling the new stiffness factor, S''_A, it is equal to $S_A \times (1 + C_{AB}) = 0.1028$.

The combined factors, $S'_A + S''_A = 2.3088$, which means that 95.55% is distributed to the vertical member and 4.45% to the horizontal member. As the total, live load, fixed-end moment, $M_A = -59\,740$ ft-lb before distribution, the actual moment at the end of Member AB after distribution will be, $-59\,740\ (1.0000 - 0.0455) = -57\,080$ ft-lb, which is the corner moment at Point A when the deck is straight. On account of the horizontal member being curved, as shown in Fig. 28, the corner moment will be less. The reduction factor[63] will be $\dfrac{17.88 + 0.75}{19.38}$, and this value times $-57\,080 = -54\,870$ ft-lb. As the corner moment is $-54\,870$, the horizontal thrust acting at the hinged support will be $\dfrac{54\,870}{17.88} = 3\,069$ lb.

Assuming, now, that the horizontal component of the reaction for the foregoing loading is 3 069 lb, then, by statics, the moment at the crown is,

$$\begin{array}{lcl} +\,90 \times 30 \times 0.5 \times 30 & = & +\,40\,500 \\ +\,0.5 \times 2\,500 \times 30 & = & +\,37\,500 \\ -\,3\,069 \times 19.375 & = & -\,59\,450 \\ \hline \text{Total}\ldots\ldots\ldots\ldots\ldots & = & +\,18\,550 \text{ ft-lb} \end{array}$$

Using a method advanced by the Portland Cement Association[64], the total positive live load moment at the crown, assuming a simply supported deck, is found to be 78 000 ft-lb. Then the difference between this moment and the negative corner moment created by the same loading is $78\,000 - 57\,080 = +\,20\,920$ ft-lb, which is the moment at the crown of the frame with a straight deck. The reduction factor[63] will be, $\dfrac{17.88 + 0.5 \times 1.50}{17.88 + 1.50} = \dfrac{18.63}{19.38}$, and this value times $20\,920 = 20\,120$ ft-lb.

The foregoing example was also solved by the writer using the "method of elastic weights", and the corner moments by that method were found to be $-55\,100$ ft-lb, and the crown moments, $+18\,310$ ft-lb. As the results by the authors' method check very closely with the method of elastic weights it shows that the former is sufficiently accurate for practical purposes.

The authors' method for the analysis of tapered members will give the desired results; but for the analysis of such a structure as the rigid-frame bridge, hinged at the supports, there are other analytical methods that are

[62] "Continuous Frames of Reinforced Concrete", by Hardy Cross and N. D. Morgan, Members, Am. Soc. C. E., N. Y., John Wiley & Sons, 1932, Fig. 38, p. 119.

[63] "Analysis of Rigid Frame Concrete Bridges", Portland Cement Assoc., Third Edition, 1935, p. 17.

[64] *Loc. cit.*, p. 24.

shorter, and if a systematic arrangement of tables is used the writer is of the opinion that the method of elastic weights will give the desired results in the shortest time.

By means of the method of elastic weights, the depth of the frame can vary along its length in any proportion, the axis can be of any shape, the end-posts need not be vertical, and, if the division points are taken sufficiently close, the results will agree very well with any of the analytical methods.

The one great disadvantage of the authors' method for the analysis of the rigid-frame bridge is that it is based on a straight-line axis for the horizontal member, and as practically all horizontal members have a curved axis a correction must be made by introducing a reduction factor which is only approximate. Another disadvantage is that, when the load is not symmetrical, there will be side-sway, and a correction must be made for that also.

Even if the method is not generally adopted, the authors should be commended for introducing a mathematical treatment which, some day, may be further elaborated, and eventually an average value of n can be selected more readily, so that the substituted I-curve will give as close results as the actual I-curve.

L. LEGENS,[65] ESQ. (by letter).[65a]—A method of analyzing structures, composed of members of non-uniform cross-section, is discussed in this paper, and the authors have indicated its application to some classical methods of analysis. The hope is expressed in conclusion that the resulting method of substitute I-curves may aid in the development of classes of structures which have been hampered in the past by mathematical difficulties of design. The use of this method, in general, is most advantageous and becomes necessary for the calculation of many basically and indeterminate systems.

For the arch fixed at the two ends, in the past, it was generally agreed, for the purpose of computation, that the reduced moment of inertia, $I' = I \cos \phi$, should be constant and equal to the moment of inertia, I_c, at the crown (see Fig. 29). Müller-Breslau[66] was one of the first to state that this assumption was not admissible, and to suggest that the arch should be computed as a tapered beam.

Assuming a parabolic arch barrel and selecting a Cartesian system of co-ordinates with the origin in O (Fig. 29), he determined the three statically indeterminate reactions, X_a, X_b, and X_c, for a concentrated load, P_m, with the general equations of elasticity.

Expressing the reduced moment of inertia, I'_n, of the end cross-sections by:

$$I'_n = \frac{1}{\alpha} I_c \qquad (200)$$

he chose for the function of the I'-curve, the formula,

$$\frac{I_c}{I'} = 1 - (1 - \alpha)\left(\frac{x}{l_1}\right)^n \qquad (201)$$

[65] Engr., Alsace-Lorraine R. R., Strasbourg, France.

[65a] Received by the Secretary December 14, 1935.

[66] "Die Graphische Statik der Baukonstruktionen", von Müller-Breslau, Band II, 2 Abteilung, Seite 556.

which is similar to Equation (1) of the paper, except that the center of ordinates for Equation (201) is at the mid-points, whereas in the paper (Equation (1)) it is at the ends. For $\alpha = 1$, Equation (201) will reduce to

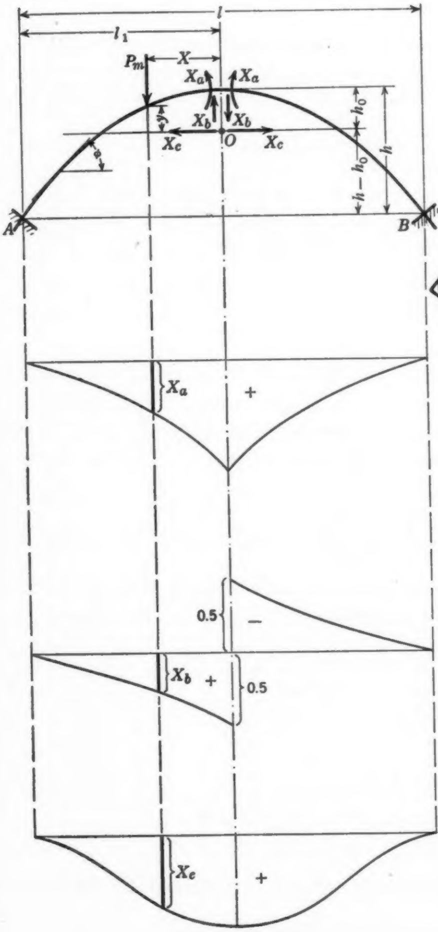

FIG. 29.

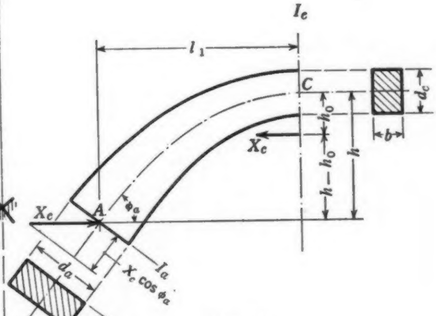

FIG. 30.

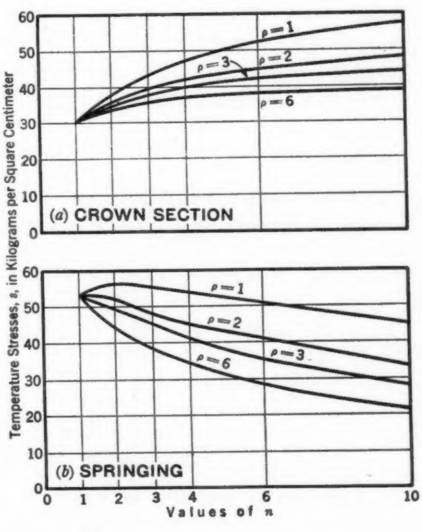

FIG. 31.—TEMPERATURE STRESSES.

$I' = I \cos \phi = I_c$, an expression which was recognized by early mathematicians; and, for the distance, h_0 (see Fig. 30), Müller-Breslau derives the expression:

$$h_0 = \left(\frac{h\ (\rho + 1)}{3\ (\rho + \alpha)}\right)\left(\frac{\rho + 3\ \alpha}{\rho + 3}\right) \dots\dots\dots\dots\dots\dots (202)$$

in which values of ρ, corresponding to n, are shown in Fig. 31.

Fig. 29 presents the influence lines for the reactions, X_a, X_b, and X_c. These values have been calculated by Müller-Breslau for various ratios, $\frac{x}{l_1}$, as shown in Table 7. The difference between the values resulting from

TABLE 7.—COMPUTATION OF REACTIONS FOR VARIOUS POSITIONS OF LOAD

$\frac{x}{l_1}$	VALUES OF X_a			VALUES OF X_b			VALUES OF X_c		
	$\alpha = 0.25$		$\alpha = 1.00$	$\alpha = 0.25$		$\alpha = 1.00$	$\alpha = 0.25$		$\alpha = 1.00$
	$\rho = 1$	$\rho = 2$		$\rho = 1$	$\rho = 2$		$\rho = 1$	$\rho = 2$	
0.0	6.0000	6.2500	7.5000	0.5000	0.5000	0.5000	0.9973	0.9950	0.9290
0.2	3.4560	3.6480	4.8000	0.3328	0.3331	0.3520	0.8941	0.8981	0.8562
0.4	1.7280	1.8180	2.7000	0.1882	0.1872	0.2160	0.6371	0.6443	0.6555
0.6	0.6720	0.6880	1.2000	0.0814	0.0789	0.1040	0.3315	0.3321	0.3805
0.8	0.1440	0.1380	0.3000	0.0191	0.0174	0.0280	0.0904	0.0864	0.1204
h_o*	......	......	......	......	......	......	3.50	3.67	5.00

* Values of h_o, in meters.

the Müller-Breslau "shape exponents", $\rho = 1$ and $\rho = 2$, is not very great; but there is considerable variation between the different values, X_a, X_b, and X_c, for $\alpha = 0.25$ or for $\alpha = 1.00$. For practical purposes, $\alpha = 0.25$ will be a good average value and it is not permissible to assume that $\alpha = 1$ (that is, to neglect the super-modulus).

It is well-known that for flat arches $\left(\frac{h}{l} \leq \frac{1}{10}\right)$, the influence of temperature is very considerable and in every case is more important than that of loads.

Referring to the general equations of elasticity, let δ = deformation; then: $\delta_{at} = 0$; $\delta_{bt} = 0$; $\delta_{ct} = 2\,\epsilon\,t\,l_1 = \epsilon\,t\,l = \Delta\,l$; $X_a = \frac{\delta_{at}}{\delta_{aa}} = 0$; $X_b = \frac{\delta_{bt}}{\delta_{bb}} = 0$; $X_c = \frac{\delta_{ct}}{\delta_{cc}}$; and,

$$X_c = \frac{\Delta\,l\,E\,I_c}{h\,l\left(\frac{1}{5}\,h\,\frac{\rho + 5\,\alpha}{\rho + 5} - \frac{1}{3}\,h_o\,\frac{\rho + 3\,\alpha}{\rho + 3}\right)} \qquad \text{..........(203)}$$

The writer prefers to use the reciprocal values introduced by Müller-Breslau. Therefore, according to Equations (200) and (201), $n = \frac{1}{\alpha}$; $I'_n = n\,I_c$; and,

$$\frac{I_c}{I'} = 1 - \frac{n - l}{n}\left(\frac{x}{l_1}\right)^{\rho} \qquad \text{..................(204)}$$

Equation (203) becomes:

$$X_c = \frac{E\,I_c}{h^2}\,\frac{\Delta\,l}{l}\,\frac{1}{\frac{1}{5}\,\frac{n\,\rho + 5}{n\,(\rho + 5)} - \frac{1}{9}\,\frac{n\,(\rho + 1)}{n\,\rho + 1}\left[\frac{n\,\rho + 3}{n\,(\rho + 3)}\right]^2} \qquad \text{....(205)}$$

Referring to Fig. 30 the unit stress, s, is obtained for the crown section, C, by the equation:

$$s = \frac{X_c}{A_c} \pm \frac{X_c\, h_0}{I_c}\, \frac{d_c}{2} \quad \text{.......... (206)}$$

in which A_c and I_c are the area and the moment of inertia and are equal, respectively, to: $A_c = b\ d_c$; and $I_c = \frac{b\ d^3_c}{12}$. When the signs, $\pm$ or $\mp$, appear, the positive refers to the intrados and the negative refers to the extrados. Finally, according to Equations (202), (205), and (206), for h_0, X_c, A_c, and I_c:

$$s = \frac{E\, \frac{\Delta\, l}{l} \left(\frac{d_c}{h}\right)^2 \frac{1}{12} \left[1 \pm \frac{2\, h}{d_c}\, \frac{\rho + 1}{\rho + 3}\, \frac{n\, \rho + 3}{n\, \rho + 1}\right]}{\frac{1}{5}\, \frac{n\, \rho + 5}{n\,(\rho + 5)} - \frac{1}{9}\, \frac{n\,(\rho + 1)}{n\, \rho + 1} \left[\frac{n\, \rho + 3}{n\,(\rho + 3)}\right]^2} \quad \text{.......... (207)}$$

For Sections A at the springing (see Fig. 30), the unit stress, s, will be:

$$s = \frac{X_c \cos \phi_a}{A_a} \mp \frac{X_c\,(h - h_0)}{I_n}\, \frac{d_a}{2} \quad \text{.......... (208)}$$

in which $I_n = \frac{b\ d^3_a}{12}$; and, $A_a = b\ d_a$. Furthermore,

$$I'_a = I_a \cos \phi_a = n\ I_c = \frac{n\, b\, d^3_c}{12} \quad \text{.......... (209)}$$

$$\frac{I_a}{I_c} = \frac{n}{\cos \phi_a}\, \frac{d^3_a}{d^3_c} \quad \text{.......... (210)}$$

$$\cos \phi_a = \frac{1}{\sqrt{1 + 4\left(\frac{h}{l}\right)^2}} \quad \text{.......... (211)}$$

and,

$$h - h_0 = \frac{2}{3}\, h\, \frac{\rho\, n\,(\rho + 4) + 3}{\rho\,[\,n\,(\rho + 3) + 1\,] + 3} \quad \text{.......... (212)}$$

from which Equation (208) becomes:

$$s = \frac{X_c \cos \phi_a}{b\ d_a} \left[1 \mp \frac{h - h_0}{b\ d^3_a} \times \frac{12\, b\, d_a}{\cos \phi_a} \times \frac{d_a}{2}\right]$$

$$= \frac{X_c \cos \phi_a}{b\, d_a} \left[1 \mp \frac{6\,(h - h_0)}{d_a \cos \phi_a}\right] \quad \text{.......... (213)}$$

and,

$$s = \frac{E\, \frac{\Delta\, l}{l}\, \frac{\cos \phi_a}{12} \left(\frac{d_c}{h}\right)^2 \left[1 \mp \frac{4\, h}{d_a}\, \frac{1}{\cos \phi_a}\, \frac{\rho\, n\,(\rho + 4) + 3}{\rho\,[\,n\,(\rho + 3) + 1\,] + 3}\right]}{\frac{1}{5}\, \frac{n\, \rho + 5}{n\,(\rho + 5)} - \frac{1}{9}\, \frac{n\,(\rho + 1)}{n\, \rho + 1} \left[\frac{n\, \rho + 3}{n\,(\rho + 3)}\right]^2} \quad \text{.......... (214)}$$

in which,

$$\frac{d_a}{d_c} = \frac{\sqrt[3]{n}}{\sqrt[3]{\cos \phi_a}} \quad \text{.....................(215)}$$

and,

$$d_a \cos \phi_a = d_c \sqrt[3]{n} \sqrt[3]{\cos^2 \phi_a} \quad \text{................(216)}$$

It follows that:

$$s = \frac{E \frac{\Delta l}{l} \frac{\cos \phi_a}{12} \left(\frac{d_c}{h}\right)^2 \frac{\sqrt[3]{\cos \phi_a}}{\sqrt[3]{n}} \left[1 \mp \frac{4\,h}{d_c} \frac{1}{\sqrt[3]{n}} \frac{1}{\sqrt[3]{\cos^2 \phi_a}} \frac{\rho\, n\,(\rho + 4) + 3}{\rho\,[n\,(\rho + 3) + 1] + 3}\right]}{\frac{1}{5} \frac{n\,\rho + 5}{n\,(\rho + 5)} - \frac{1}{9} \frac{n\,(\rho + 1)}{n\,(\rho + 1)} \left[\frac{n\,\rho + 3}{n\,(\rho + 3)}\right]^2} \quad \text{........(217)}$$

For flat arches one may assume that $\sqrt[3]{\cos \phi_a}$ varies as 1 and $\sqrt[3]{\cos^2 \phi_a}$ varies as 1, in which case, Equation (217) becomes:

$$s \;\infty\; \frac{E \frac{\Delta l}{l} \left(\frac{d_c}{h}\right)^2 \frac{1}{12 \sqrt[3]{n}} \frac{1}{\sqrt{1 + 4\left(\frac{h}{l}\right)^2}} \left[1 \mp \frac{4\,h}{d_c} \frac{1}{\sqrt[3]{n}} \frac{\rho\, n\,(\rho + 4) + 3}{\rho\,[n\,(\rho + 3) + 1] + 3}\right]}{\frac{1}{5} \frac{n\,\rho + 5}{n\,(\rho + 5)} - \frac{1}{9} \frac{n\,(\rho + 1)}{n\,\rho + 1} \left[\frac{n\,\rho + 3}{n\,(\rho + 3)}\right]^2}$$

Finally, for the crown section:

$$s = E\, \frac{\Delta l}{l} \left(\frac{d_c}{h}\right) \frac{1}{6}\, h_3\,(\rho, n) \left[\frac{1}{2} \left(\frac{d_c}{h}\right) \mp h_4\,(\rho, n)\right] \quad \text{....... (218)}$$

and, for the section at the springing:

$$s \;\infty\; E\, \frac{\Delta l}{l} \left(\frac{d_c}{h}\right) \frac{1}{3} \frac{1}{\sqrt{1 + 4\left(\frac{h}{l}\right)^2}}\, h_1\,(\rho, n) \left[\frac{1}{4} \left(\frac{d_c}{h}\right) \pm h_2\,(\rho, n)\right] \quad \text{... (219)}$$

in which:

$$h_1\,(\rho, n) = \frac{1}{\frac{1}{5} \frac{n\,\rho + 5}{n\,(\rho + 5)} - \frac{1}{9} \frac{n\,(\rho + 1)}{n\,\rho + 1} \left[\frac{n\,\rho + 3}{n\,(\rho + 3)}\right]^2} \frac{1}{\sqrt[3]{n}} \quad \text{...(220)}$$

$$h_2\,(\rho, n) = \frac{\rho\, n\,(\rho + 4) + 3}{\rho\,[n\,(\rho + 3) + 1] + 3} \frac{1}{\sqrt[3]{n}} \quad \text{...........(221)}$$

$$h_3\,(o, n) = \frac{1}{\frac{1}{5} \frac{n\,\rho + 5}{n\,(\rho + 5)} - \frac{1}{9} \frac{n\,(\rho + 1)}{n\,\rho + 1} \left[\frac{n\,\rho + 3}{n\,(\rho + 3)}\right]^2}$$

$$= \sqrt[3]{n}\, h_1\,(\rho, n) \quad \text{.................................(222)}$$

and,

$$h_4(\rho, n) = \frac{\rho + 1}{\rho + 3} \frac{n\rho + 3}{n\rho + 1} \quad \text{.................(223)}$$

Values of h are listed in Table 8. The maximum stresses are found for the intrados at the crown and for the extrados at both springing lines.

TABLE 8.—VALUES OF h (ρ, n) IN EQUATIONS (220) TO (223), INCLUSIVE

Values of n	For Values of ρ Equal to:				For Values of ρ Equal to:			
	1	2	3	6	1	2	3	6
	(a) Values of h_1 (Equation (220))				(c) Values of h_3 (Equation (222))			
1	11.25	11.25	11.25	11.25	11.25	11.25	11.25	11.25
2	13.50	12.55	12.01	11.11	17.01	15.81	15.14	14.00
3	14.68	13.00	12.08	10.65	21.18	18.75	17.42	15.36
4	15.38	13.11	11.92	10.19	24.41	20.82	18.93	16.18
6	16.03	12.95	11.45	9.41	29.13	23.53	20.80	17.11
10	16.20	12.27	10.52	8.33	34.90	26.43	22.66	17.95
	(b) Values of h_2 (Equation (221))				(d) Values of h_4 (Equation (223))			
1	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2	0.86	0.86	0.85	0.83	0.83	0.84	0.86	0.90
3	0.78	0.77	0.77	0.74	0.75	0.77	0.80	0.86
4	0.72	0.71	0.70	0.68	0.70	0.73	0.77	0.84
6	0.65	0.64	0.62	0.61	0.64	0.69	0.74	0.82
10	0.56	0.54	0.53	0.51	0.59	0.66	0.71	0.80

The writer has computed these stresses for a concrete arch bridge, 74 m (242.8 ft) long and 6.75 m (22.1 ft) high for super-moduli of $n = 1$, 2, 3, 4, 6, and 10, and the shape exponents, $\rho = 1$, 2, 3, and 6. They are shown in Fig. 31. With increasing values of n and ρ the stresses decrease at the springing, increase at the crown, and are almost equal at both sections for $n = 5$ and $\rho = 2$ which are good average values as determined for a considerable number of actual bridges.

For practical purposes it may be sufficient to compute the influence of temperature by the old formulas (assuming $n = 1$), the actual stresses being always less than those calculated.

In conclusion, the writer wishes to express his appreciation of the clear method described in this paper. It permits a rapid analysis of structures with variable moments of inertia. Although Müller-Breslau anticipated the method proposed by the authors, he did not apply it in such a general manner. Of course, the equations presented by the writer could be easily transformed to make them agree with the equation for the moment of inertia of tapered members presented by Messrs. Weiskopf and Pickworth.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

STABLE CHANNELS IN ERODIBLE MATERIAL

Discussion

BY MESSRS. V. V. TCHIKOFF, AND W. M. GRIFFITH

V. V. TCHIKOFF,[25] M. AM. SOC. C. E. (by letter).[25a]—An excellent analysis of a complex problem is contained in this paper, and the author should be congratulated for his collection of the data accompanying it. The writer approves of most of the propositions suggested in the paper; they appear to be logical and convincing. The special consideration of the stability of the banks and the presentation of the cross-sections with the "isovels" are the most important additions to the study of the problem (see Fig. 3[25b]).

Isovels.—The purpose of a number of the rectangular cross-sections with the isovels presented by the author is to show "that high velocities extend closer toward the sides in the narrow, deep cross-sections than in the broad, shallow ones." However, an examination of Fig. 3 indicates that all cross-sections except one ($\frac{B}{d} = 0.60$, which is almost never used in practice), have the same isovel, 0.80, nearest the sides. The author's statement is perhaps more applicable to the bottom velocities, although there is a discrepancy in their distribution. The isovel, 0.80, for instance (shown in the sections with $\frac{B}{d} = 5.95$ and $\frac{B}{d} = 9.20$), disappears at the bottom of the intermediate section in which $\frac{B}{d} = 7.48$; but a more important observation is that both the side and the bottom isovels, being equal to 0.80 in most cases, do not vary much, especially if the section with $\frac{B}{d} = 0.60$ (which is not a very practical one) is disregarded.

NOTE.—The paper by E. W. Lane, M. Am. Soc. C. E., was published in November, 1935, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: February, 1936, by R. C. Johnson, M. Am. Soc. C. E.; April, 1936, by Messrs. E. S. Lindley, J. C. Stevens, C. R. Pettis, Harry F. Blaney, and Sigurd Eliassen; and May, 1936, by Messrs. R. E. Ballester, and Gerald Lacey.

[25] Cons. Engr., Washington, D. C.

[25a] Received by the Secretary April 10, 1936.

[25b] *Proceedings*, Am. Soc. C. E., November, 1935, p. 1320, in Fig. 3, "Note", read "d = depth" for "D = depth."

It is well known that the mean velocity and the corresponding side and bottom velocities vary considerably in stable channels of the various sizes carrying the same sediment. Kennedy's stable sections[26] serve as an example; the mean velocity in the smallest section is 1.31 ft per sec, whereas, in the largest section, the velocity is 2.86 ft per sec. This variation is still more noticeable in Lindley's sections[27], the mean velocities ranging from 1.03 to 3.61 ft per sec, or about 350 per cent. Table 7 further illustrates how the

TABLE 7.—DEPTH AND VELOCITY IN RECTANGULAR STABLE CHANNELS, CORRESPONDING TO VARIOUS WIDTH-DEPTH RATIOS

Item No.	Width-depth ratio, $\frac{B}{d} = r$	HYDRAULIC NUMBER, $N = 40$				HYDRAULIC NUMBER, $N = 20$			
		Kinetic Factor, $K = 0.04$		Kinetic Factor, $K = 0.16$		Kinetic Factor, $K = 0.04$		Kinetic Factor, $K = 0.16$	
		Depth, d, in feet	Velocity, V, in feet per second	Depth, d, in feet	Velocity, V, in feet per second	Depth, d, in feet	Velocity, V, in feet per second	Depth, d, in feet	Velocity, V, in feet per second
1.....	3.20	2.32	1.36	2.32	2.71	9.28	2.71	9.28	5.42
2.....	5.95	3.02	1.71	3.02	3.41	12.08	3.41	12.08	6.82
3.....	11.00	5.56	2.46	5.56	4.92	22.24	4.92	22.24	9.85
4.....	20.00	13.00	3.90	13.00	7.80	52.00	7.80	52.00	15.60

velocities are augmented with the growing of the width-bed ratio. These changes of velocity are incomparably larger than the rate of variation, if any, of the isovels near the side and bottom in the cross-section presented by the author.

Velocity Gradient.—The approximate value of the velocity gradient for the middle vertical can be calculated by dividing the difference of the two neighboring velocities or isovels by the vertical distance between them. This distance, however, scarcely changes for the low isovels (0.80, 0.90, and 1.00) in the greater part of the sections shown in the paper. There is a minor irregularity in the position of the isovel, 1.10, and a more noticeable one in that of the isovel, 1.20. The maximum velocity gradient is usually found in the low parts of the cross-sections, but it remains practically constant in the cross-sections of the paper.

"Pressure Gradient."—The action of the sides of a channel is mainly typified by the depression of the filament of maximum velocity. This is due to the existence of the transverse current in the flow, which was observed and experimentally demonstrated by Dr. A. H. Gibson[28]. The double-spiral-motion theory and the existence of the cross-currents with the direction of their flow toward, and up along, the sides of a channel could explain, reasonably well, such observed phenomena as the undermining of the low parts of the sides of channels, the rounding up of the angles between the bottom and sides, and the silting of the upper parts of the sides.

[26] "Stable Channels in Alluvium", by Gerald Lacey, p. 286.

[27] *Loc. cit.*, p. 287.

[28] "On the Depression of the Filament of Maximum Velocity in a Stream Flowing Through an Open Channel", by A. H. Gibson, *Proceedings*, Royal Soc. of London, Vol. 82-A, 1909.

The sum of the potential and kinetic energy is constant for all the points of the stream filament that lies on the same horizontal plane. Applying this condition, the "pressure gradient" that causes the transverse current may be expressed by the formula:

$$\frac{V^2_c - V^2_s}{2g\,\dfrac{B}{2}} = \frac{V^2_c - V^2_s}{g B} \quad\text{..................(25)}$$

in which V_c = a velocity at the center of the cross-section; and V_s = a velocity on the same horizontal plane at the side of a channel. The isovels do not vary much in the cross-section in Fig. 3, but the value of B changes considerably. The broader the channel, the smaller the "pressure gradient" and the transverse currents, and the more stable are its banks. The effect of the sides of a channel is to produce the curvilinear movement of the flow, the cross-sectional profile of the water surface usually being a curve. As a result, the actual vertical pressure differs from the hydrostatic-pressure triangle. The difference depends upon the rate of changes of the pressure heads corresponding to the near-by velocities.

The theoretical equation for the vertical velocity curve is a parabola of the following type:

$$V = a\,y^2 + b\,y + c \quad\text{....................(26)}$$

The equation for the corresponding pressure heads will be,

$$h_p = \frac{1}{2\,g}\,(a\,y^2 + b\,y + c)^2 \quad\text{................(27)}$$

and the rate of the changes of these pressure heads, S_p (= pressure gradient), is,

$$S_p = \frac{h_p}{dy} = \frac{V}{g}\,S_v \quad\text{....................(28)}$$

in which S_v is the velocity gradient. Due to the combined effect of the isovels and the velocity gradient changes, both leaning in the same direction, the variation of the pressure gradient in Fig. 3 is greater than the change of each component, but even this double effect cannot exppain the decidedly different property of the sections in Fig. 3 in regard to their relative stability. The reason for this, in the writer's opinion, is that each individual cross-section will fit a certain set of the hydraulic conditions upon which its stability depends. The condition for stability remaining unchanged, the width-depth ratio is determined according to the discharge of flow. For explanation of this statement, three general hydraulic factors should be considered: The kinetic factor; the coefficient of hydraulic similtude; and the hydraulic number; which have been introduced by the writer elsewhere.

Kinetic Factor.—The kinetic factor[29] is equal to,

$$K = \frac{V^2_m}{g\,R} \quad\text{........................(29)}$$

[29] *Transactions*, Am. Soc. C. E., Vol. 100 (1935), p. 171, (Equation (73); also, pp. 852–858.

It is a measure of the kineticity of flow, or turbulence. Equation (29) is similar to Mr. Lacey's formula expressed as Equation (7) in the paper, except that the kinetic factor has a definite physical significance. Equation (29) may be rewritten in the form:

$$V_m = (g\,K)^{0.5}\,R^{0.5} \qquad (30)$$

"Sediment Equivalent."—Lacey correlates his silting factor to the diameter of the bed material by Equation (8) which, expressed in terms of the kinetic factor, is as follows:

$$D = 8.6\,K \qquad (31)$$

Lacey gave no precise indication of the classification of the sediment; nor has he considered the quantity of the material transported. It should be noted, however, that both Kennedy and Lacey connected their formulas to the sediment taken from the bed of canals or rivers.

By designating the hydraulic value of a particle by ω (the settling velocity in the water) and the concentration of the sediment by η (percentage by weight), the work per second (power) produced by the settlement in the water of a given sample of sediment may be termed the "sediment equivalent" and expressed by:

$$E = \frac{\rho_s - \rho_w}{\rho_s} \sum (\omega\,\eta) \qquad (32)$$

in which ρ_s is the density of the particles; ρ_w is the density of the water; and $\Sigma\,(\omega\eta)$ = the sum of $\omega\eta$ for each class of particles according to mechanical analysis. If ω is measured in feet per second and η in pounds (in 100 lb of water), E is in foot-pound units. For computing E it is convenient to use the mean hydraulic value of a sample, ω_m, which is equal to $\frac{\Sigma\,(\omega\,\eta)}{\eta_t}$, in which η_t is the total concentration. If each class of particles in a mechanical

TABLE 8.—MECHANICAL ANALYSES, MEAN HYDRAULIC VALUES, AND SEDIMENT EQUIVALENTS, FOR TWO CANALS IN IMPERIAL VALLEY, CALIFORNIA

Canal	PERCENTAGE OF SILT PASSING A GIVEN SIEVE (SIEVE NUMBERS IN MESHES PER INCH)							
	10	20	40	60†	80	100	200	300
Hydraulic value, ω_a*	0.361	0.191	0.111	0.073	0.054	0.033	0.013	0.0036
Alamo..........	0	0	0.91	....	5.35	16.34	15.38	62.02
Brawley.........	0	0	0.54	....	8.86	13.58	17.38	59.64

Canal	Location	Depth to sample, in feet‡	Mean hydraulic value, ω_m, in feet per second	Concentration of sediment η (percentage of silt by weight)	Sediment equivalent, E	Kinetic factor, K	Ratio, a, $\frac{E}{K^2}$
Alamo...........	Hanlon §.	9.8	0.014	0.486	0.0043	0.075	0.76
Brawley..........	Central..	4.2	0.015	0.542	0.0051	0.074	0.93

* Average hydraulic value of particles, corresponding to the sieve analysis, in feet per second. † 60-mesh sieve omitted. ‡ 0.2 ft above the canal bed, in each case. § 120 ft from the east bank.

analysis were indicated by the percentage, ρ, of the total concentration, the mean hydraulic value would be equal to $\frac{\Sigma (\omega \rho)}{100}$ (see Table 8). The "'sediment equivalent" may be used as a criterion for the comparison of sediments, and it is rightly comparable to the kinetic factor.

Quantity and Quality of Sediment.—If two samples of the sediment, each consisting of only one size of particles of the same density, have the same value, E, the following equation is applicable: $\omega_1 \eta_1 = \omega_2 \eta_2$; or,

$$\frac{\eta_1}{\eta_2} = \frac{\omega_2}{\omega_1} \qquad (33)$$

Equation (33) means that the concentration of the sediment varies in reverse proportion to the hydraulic values of the particles. This relation appears to reveal the approximate general law of the transportation of sediment in suspension. The law may be expressed in general terms:

$$\frac{\eta'_t}{\eta''_t} = \frac{\omega''_m}{\omega'_m} \qquad (34)$$

That is, if the "sediment equivalents" are equal, the total concentration of the sediment transported in suspension varies in reverse proportion to the mean hydraulic value of the samples. There is a certain limit for the maximum hydraulic value, $\omega_{max.}$, of the particles that are found in the samples, which have the same value as the "sediment equivalent."

Kinetic Factor and "Sediment Equivalent."—The condition for the transportation of the sedimentary material near the bed in a stable channel in erodible material is of primary importance. The suspended material and bed-load merge into each other near the bed. This is especially true concerning the flow which is heavily charged with fine sediment, as the bed-load differs but little from that carried in suspension. Even if the sand were presented in a noticeable quantity, the same merging effect would occur and the coarse particles are often found at a relatively high distance from the bottom. The kineticity of the flow is directly related to the transportation of the sedimentary material near the bed, especially if this material is not of a coarse nature. Since the kinetic factor is a measure of turbulence and the sediment equivalent serves as a silt criterion, they should be related to each other. On the basis of certain considerations, it appears that their relation may be stated by the following equation:

$$E = \alpha K^2 \qquad (35)$$

in which α is a coefficient. The kinetic factor, K, is a dimensionless number, and, therefore, α is in foot-pound units. The determination of α for the two Imperial Valley canals[30] is demonstrated in Table 8. The average hydraulic value for each class of particles (see Table 8), more or less corresponds to

[30] "Silt in the Colorado River and Its Relation to Irrigation", by the late Samuel Fortier, M. Am. Soc. C. E., and Harry F. Blaney, M. Am. Soc. C. E., Tables 55 and 56, p. 75.

the Hazen formula[31]. One-half the hydraulic value of the particles corresponding to the 300-mesh sieve is assumed for the sediment passed through this sieve. Although this class of sediment constitutes a considerable portion of the total concentration, its work during the settlement is relatively small. The silt samples shown in Table 8 are the nearest to the bed of the canals, being in both cases 0.2 ft above the bottom. With the density of the sediment assumed as 2.65, the values of the coefficient, α, are equal to 0.76 and 0.93. The lack of information in regard to the stability, and especially in regard to "degree of concentration" of the sediment, makes it difficult to appreciate these values.

Degree of Concentration.—Three states of the flow carrying the sediment are distinguishable: (*a*) The over-loaded flow; (*b*) the normally loaded flow; and (*c*) the sub-normally loaded flow. The criterion for this general classification is the relation of the sediment load to the kineticity of the flow.

The flow, with a certain kineticity, will only carry the definite sediment load, provided, of course, the necessary quantity of sediment is supplied. This will be a normal sediment-loaded flow. Any increase in the supply of the sediment will over-load the flow with the result that the extra supply will drop. On the contrary, the under-loaded flow tends to pick up the material from the bottom and will scour the channel, depending upon the bed resistance. If the sediment is non-cohesive, there would be a tendency toward the normally loaded flow.

The value of the coefficient, α, depends upon the degree of sediment concentration. For the normally loaded flowing water in a stable channel, it seems possible to suggest that the value of α would be equal approximately to a unit if the sediment at the bottom and the kinetic factor are considered. It is difficult to obtain a representative sample of the sediment transported at the bottom, due to the merging situation. If the sediment from the bottom is considered, the nominal sediment concentration may be computed from Equation (35) on the basis of mechanical analysis of the bed material and under the assumption that $\alpha = 1$.

Sediment Concentration at the Bottom.—Twelve mechanical analyses of bed sediment in the Imperial Valley canals for various stations, at distances of 48 to 104 miles from the river, are given by Messrs. Fortier and Blaney[32]. The result of the calculation of the mean hydraulic values, ω_m, for these samples ranges from 0.023 to 0.047. However, the several highest values of ω_m have been influenced by the local conditions, as the wind-blown sand changes the character of the bed deposit. The average value for all twelve stations is 0.0324, and it is reduced to 0.03 if the three highest values are disregarded. The value of the kinetic factor for the Imperial Valley canals is analyzed in another part of this discussion (see "Stable Canals of Imperial Valley Project"). Its average value is equal to 0.117. Substituting this value into Equation (35) and assuming $\rho_s = 2.65$, the concentration, η, is 0.73 (percentage by weight). This number corresponds to the mean hydraulic value

[31] "Distribution of Silt in Open Channels", by J. E. Christianson, *Transactions*, Am. Geophysical Union, 1935, Pt. II, p. 481.

[32] "Silt in the Colorado River and Its Relation to Irrigation", by the late Samuel Fortier, M. Am. Soc. C. E., and Harry F. Blaney, M. Am. Soc. C. E., Table 49, p. 72.

of 0.03, and the concentration will vary directly with ω_m and K. More information on this point is needed, but the writer believes that the foregoing value is more or less comparable to the concentration of the normally sediment-loaded flow in the Imperial Valley canals. It should be noted that the foregoing consideration refers to the non-silting non-scouring canals.

With reference to the canals of British India, the writer has in his possession[33] the analysis of the sediment from the bed of the Sutlej River which was considered by Kennedy as one of the typical analyses corresponding to his formula,

$$V_0 = 0.84\, d^{0.64} \qquad (36)$$

This analysis is as follows:

Hydraulic value, ω, in feet per second.	Percentage by volume	Hydraulic value, ω, in feet per second.	Percentage by volume
0.00 to 0.05	11	0.20 to 0.30	7
0.05 to 0.10	43	0.30 to 0.40	2
0.10 to 0.15	25	0.40 to 0.50	1
0.15 to 0.20	11		

The mean hydraulic value of this sample is equal to 0.114, or approximately four times larger than the bed sediment of the Imperial Valley canals. Substituting into Equation (35) and taking $K = 0.0415$ (for Equation (36)[34]), $\rho_s = 2.65$ and $\alpha = 1$, the concentration is approximately equal to 0.0242% by weight, or approximately a ratio of 1 to 4 130 by weight. This quantity is comparable to the sediment concentration of the Sirhind Canal which takes its supply from the Sutlej River. The concentration in this canal ranges from 1 in 3 300 to 1 in 9 000 by volume[33]. The highest concentration causes the silting, whereas the scouring was observed at the time of low concentration. Therefore, it may be assumed that the average (approximately 1 to 4 000 by volume) is the normal concentration that is more or less comparable to 1 in 4 130 by weight. The foregoing calculations are approximate, but the purpose of their presentation is to indicate the existence of a quantitative relation between the kinetic factor and the sediment at the bottom of the stable channels. It is also possible to show that the kinetic factor and the "pressure gradient" near the bottom are directly related.

Coefficient of Hydraulic Similitude and Hydraulic Number.—The knowledge of the kinetic factor is not sufficient for determining the dimensions of a stable channel. In addition, one of the two other factors should be known: Either the coefficient of hydraulic similitude, denoted as Y:

$$Y = \frac{Q}{R^3} \qquad (37)$$

or, the hydraulic number, N, related to Y by the equation:

$$Y = N\, K^{0.5} \qquad (38)$$

[33] The data are taken from the writer's publication on "Silting of the Irrigation Canals" (Petrograd, 1915), which was the result of his trip to British India.

[34] *Transactions*, Am. Soc. C. E., Vol. 100 (1935), p. 853.

The coefficient of hydraulic similitude defines the condition that the discharges of similar stable channels are in proportion to their hydraulic radii, raised to the third power. The hydraulic number gives the relation between the other two factors. Representing area of cross-section of a channel by $A = \beta R^2$ and substituting the values of K and Y from Equations (29) and (37), Equation (38) gives:

$$N = g^{0.5} \frac{\beta}{R^{0.5}} \qquad (39)$$

in which β is a "hydraulic" shape factor. The writer has presented the values of these factors elsewhere[35]. The rectangular cross-sections in this paper furnish an opportunity to examine this type of channel by applying the aforementioned factors. For a stable channel of rectangular form, the relation between depth, hydraulic number, and width-depth ratio, $r = \frac{B}{d}$, is expressed by the following equation:

$$d = \frac{g\,(r + 2)^5}{N^2\, r^3} \qquad (40)$$

which indicates that the depth in the rectangular non-silting non-scouring channels depends upon only the hydraulic number and the width-depth ratio. Equation (40) is presented by a series of diagrams on Fig. 9. by substituting for r, its equivalent, $\frac{B}{d}$, Equation (40) may be rewritten in a form, giving the direct relation between the depth and the width.

The paper includes two diagrams, Figs. 1 and 2, which give the velocity depth and the bed-width-depth relation for numerous formulas. It is natural that the critical velocity-depth relations (Fig. 1), being contingent on the kinetic factor and the character of the sediment, should vary considerably. The formulas of the type of Equation (1) are analyzed elsewhere in this discussion.

In regard to the equations for width-depth relation (Fig. 2), the author, taking the width corresponding to the 5-ft depth for two formulas, the Molesworth-Yenidunia formula and the Lindley formula, remarks that a ratio of maximum to minimum is of 781 per cent. It is also natural that the bed-width corresponding to the same depth must vary considerably, as it is dependent upon the hydraulic number, as illustrated by Fig. 9.

The width-depth formulas and the velocity-depth equations should be studied together if they were developed for the same conditions. Two previously mentioned formulas, the Molesworth-Yenidunia for $S = 0.00010$ and 0.00007, and the Lindley formula, are drawn on Fig. 9. Their positions are quite consistent with the remainder of the curves. An hydraulic number of about 20 is rather typical of the Egyptian canals, whereas that of the Lindley formula lies between 35 and 45. However, it should be repeated

[35] *Transactions*, Am. Soc. C. E., Vol. 100 (1935), pp. 853–855, Tables 5, 7, and 8.

that the curves in Fig. 9 are for rectangular channels. Their application to other forms of channels, therefore, has some limitations.

Stability of Rectangular Cross-Sections.—Assuming the values of the kinetic factor, the hydraulic number, and the width-depth ratio, the other

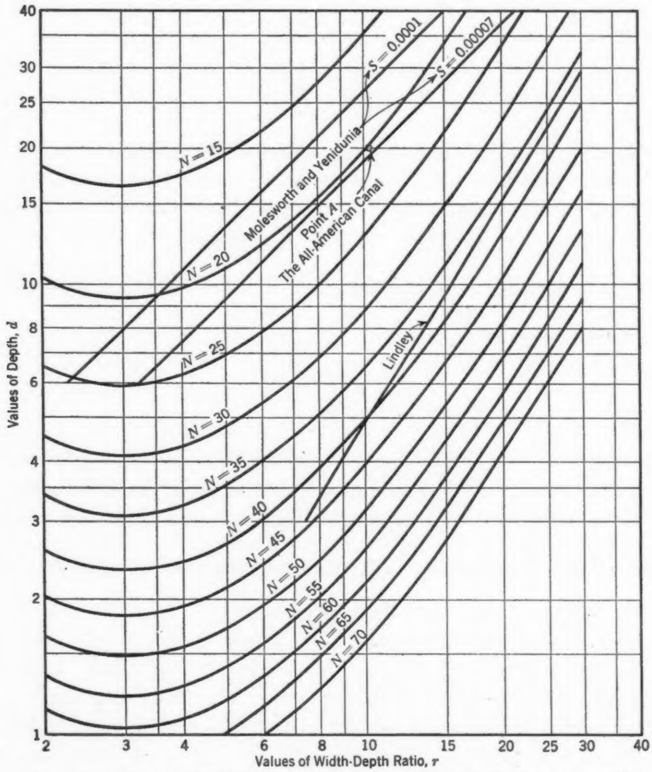

FIG. 9.—DEPTH, WIDTH-DEPTH, RATIO, AND HYDRAULIC NUMBER FOR RECTANGULAR, NON-SILTING, NON-SCOURING CHANNELS.

hydraulic elements can be calculated for the rectangular cross-sections. For the computation of the corresponding discharge, the following formula applies:

$$Q = g^3 \frac{K^{0.5}}{N^5} \left[\frac{(r + 2)^2}{r} \right]^6 \qquad (41)$$

Table 7 gives the depth and velocity for the various values of K, H, and r. It is obvious from Equation (30) that the velocity changes in proportion to the square root of the kinetic factor. For the same width-bed ratio, the influence of the hydraulic number upon the velocity is that the latter varies in reverse ratio to the hydraulic number.

The width-bed ratio of the cross-sections in Fig. 3 is given in Item Nos. 1, 2, and 3, of Table 7. Considerable variations of the velocity should be noted for the cross-section that has the same width-bed ratio. For example, the

range of velocities in cross-section, $r = 5.95$, is from 1.71 to 6.82 ft per sec, depending upon the values of K and N. The examination of the velocities for the same value of K and N (vertical columns) also shows that they vary considerably They increase with the growing value of the width-depth ratio. There is a practical limit for the increase in the velocity, mainly because of irregularities in the bottom of the canal. However, for a given value of K, N, and Q, the only section of a definite width-depth ratio will be stable, at least theoretically, for the non-cohesive sediment.

For an illustration of the manner in which the foregoing relations could explain the condition of unstable channels, an example is presented. Among Lacey's data[36] there is a cross-section of the Upper Bari Doab Canal, which has "a velocity approximating Kennedy's regime velocity, yet it had scoured its bank for 20 yars past." The actual section of this canal is shown in the solid line on Fig 10. A "normal" section is drawn in dotted lines, and both

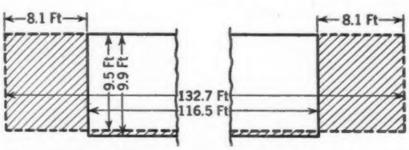

FIG. 10.—ACTUAL AND NORMAL CROSS-SECTIONS, UPPER BARI DOAB CANAL, MAIN BRANCH.

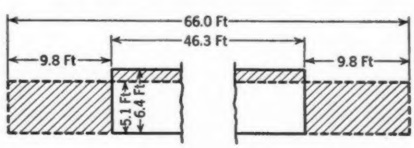

FIG. 11.—VARIATION OF NON-SILTING, NON-SCOURING SECTIONS, EAST HIGHLINE CANAL, AT B HEADING, IMPERIAL VALLEY, CALIFORNIA.

sections are assumed to be of the rectangular form. The "normal" section is determined, using the same discharge and assuming that $K = 0.0415$ and $N = 35.3$, the values that are in Kennedy's formula for this project. Comparing the two sections, the canal scoured to the depth of 9.9 ft instead of the "normal" depth of 9.5 ft, but its "normal" width should be 16.2 ft wider. This condition naturally causes scouring of the banks, which apparently are more resistant than the "normal" ones should be.

Formulas of the Type of Equation (1).—This type of formula has been used extensively for the expression of the relation between the critical velocity and depth, as is well illustrated by Table 1. The same kind of formula, however, can be developed from the data of Table 7. For example, for a velocity and depth corresponding to $K = 0.04$ and $N = 40$, four variations of Equation (1) may be written with two unknowns, C and n. Numbering these equations (A), (B), (C), (D), in the order of the bed-depth ratio values in Table 7, there are six combinations of these equations for finding the unknown value of the exponent, n. The results are as follows:

Combinations of Equations:	Values of n:	Combinations of Equations:	Values of n:
(A) and (B)	0.87	(B) and (C)	0.60
(A) and (C)	0.68	(B) and (D)	0.57
(A) and (D)	0.61	(C) and (D)	0.54

An examination of these and similar data for the other values of K and N, leads to several interesting conclusions.

[36] "Stable Channels in Alluvium", by Gerald Lacey, p. 289.

The exponent, n, is not influenced much by the values of K and N; it depends primarily upon the width-depth ratio. The larger this ratio, the smaller the value of n will be. Its limit is equal to 0.50 for a channel of indefinite width. The range of the width-depth ratio in the canals that have been observed for the development of a specific formula of the type discussed, mainly if not primarily determines the value of n. The more channels with small width-depth ratios that are included, the greater will be the value of n. For instance, the width-depth ratio in the twenty-two cross-sections of the Bari Doab Canal which served for the development of Kennedy's formula, with $n = 0.64$, ranges from 3.5 to 12.7, the group from 3.5 to 5.0 constituting more than 40 per cent. For the data given previously, 0.66 is the average of all six values of n, and this average would be 0.57 if Equation (A) pertainng to the lowest value, $r = 3.20$, is excluded. Assuming 0.57 for the exponent, n, the values of Coefficient C would be as follows: 0.85, 0.92, and 0.93, the mean being 0.90. Consequently, the form of Equation (1) becomes:

$$V_0 = 0.90\ d^{0.57} \quad \text{(42)}$$

The value of C for the others, K and N (Table 7) in the formulas:

$$V = C\,R^{0.5} \quad \text{(43)}$$

and

$$V = C\,d^{0.57} \quad \text{(44)}$$

is given in Table 9.

TABLE 9.—COEFFICIENT, C, IN EQUATIONS FOR VELOCITY IN RECTANGULAR STABLE CHANNELS

Description	HYDRAULIC NUMBER, $N = 40$		HYDRAULIC NUMBER, $N = 20$	
	Kinetic factor, $K = 0.04$	Kinetic factor, $K = 0.16$	Kinetic factor, $K = 0.04$	Kinetic factor, $K = 0.16$
Equation (43) . .	1.135	2.27	1.135	2.27
Equation (44) . .	0.90	1.80	0.82	1.64

It is of interest to mention that Lorenz G. Straub, Assoc. M. Am. Soc. C. E., found the exponent to be equal to 0.56 for the non-silting, non-eroding condition in his study on the bed-sediment transportation.[37]

Unlike the exponent, n, which is practically independent of the kinetic factor and the hydraulic number, the value of the coefficient, C, depends upon both of them. For the same value of n and r (rectangular channels considered), the ratio of the C-values may be expressed by the equation:

$$\frac{C_2}{C_1} = \left(\frac{K_2}{K_1}\right)^{0.50} \left(\frac{N_2}{N_1}\right)^{2n-1} \quad \text{(45)}$$

[37] "Hydraulic and Sedimentary Characteristics of Rivers", by Lorenz G. Straub, Assoc. M. Am. Soc. C. E., *Transactions*, Am. Geophysical Union, April, 1932.

A similar type of equation may be developed by using one-half, or the entire width, of the channels. If d in Equation (1), is equal to $\frac{B}{2}$, the coefficient, C, will change slightly, but the exponent, n, will be noticeably less than in an equation of the usual type.

Stable Canals of Imperial Valley Project.—William T. Collings, Jr., M. Am. Soc. C. E., has published the hydraulic elements for six canals of the Imperial Valley Project[38]. His data have been used by the writer in preparing Table 10, giving the general hydraulic factors for these canals. For the first four canals (Items Nos. 1, 2, 3, and 4), all the sections but one have been taken, which are considered by Mr. Collings as non-silting, non-scouring sections. Average values for the kinetic factor, coefficient of hydraulic similitude, and hydraulic number are presented in Columns (8), (9), and (10), of Table 10. These data permit the division of these six canals into two groups.

TABLE 10.—General Hydraulic Factors, Canals in the Imperial Valley, California

Item No.	Canal	Location	Discharge, Q, in Cubic Feet per Second, Ranging:		Mean Velocity V_m, in Feet per Second, Ranging:		Hydraulic slope, S	Mean kinetic factor, K_m	Mean coefficient of hydraulic similitude, Y_m	Mean hydraulic number, N_m
			From	To	From	To				
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
1	Alamo	Alamo Mocho, Meter Station, in Mexico	2 900	3 600	4.6	5.0	0.000255	0.127	19.6	54.9
2	East Highline	"B" Heading	705	1 300	3.5	4.4	0.000322	0.108	12.1	37.0
3	Central Main Canal	Alamo Mocho Meter Station	500	750	3.5	4.0	0.000440	0.120	13.1	37.7
4	Briar	International Boundary Line	180	230	2.9	3.1	0.000382	0.115	15.3	44.5
5	West Side	Boundary Meter Station	400	750	2.8	3.3	0.000525	0.063	5.7	22.7
6	West Side Main Canal	Drain Meter Station, in California	450	650	2.9	3.4	0.000380	0.064	4.9	19.3

The first group consists of the four canals (Items Nos. 1, 2, 3, and 4) having practically the same kinetic factor, which ranges from 0.108 to 0.127, the average being 0.117 ± 8 per cent. The two remaining canals (Items Nos. 5 and 6) belong to the second group, having practically the same kinetic factor (0.063), but it appears that both canals are not stable and the material composing their cross-sections differs greatly from that made by the sediments of the Imperial Valley Canal. In regard to Item No. 5, Table 10 (West Side, Main Canal, Boundary Meter Station, California), Mr. Collings made a remark that "the sides and bottom show a hard and tight clay, and an eroded rough surface." The other canal (Item No. 6, Table 10) is called "self-maintaining" by Mr. Collings: "What silting may take place at low heads is relieved when these heads increase, and no cutting of the banks takes place at maximum heads." It should be noted that the hydraulic number is about

[38] *Transactions*, Am. Soc. C. E., Vol. 99 (1934), p. 549.

20 in these two canals, or approximately one-half as much as in the canals of the first group. These values of kinetic factor and hydraulic number are comparable with those that have been found by the writer in eroding drainage channels.[39] Hence, the writer is inclined to believe that the second group belongs rather to the canals scoured in "earth", whereas the first group is typical for the Imperial Valley canals, which are built up, or strongly affected, by the sediment transported by the system. Accepting the kinetic factor, $K = 0.117$, as a typical one for this project, the equation for the mean critical velocity will take the form:

$$V_0 = 1.93\ R^{0.5} \qquad (46)$$

Equation (46) covers a considerable range of discharges, from 180 to 3 600 cu ft per sec, and of velocities, from 2.9 to 5.0 ft per sec.

Influence of Fine Particles.—For estimating the cohesive force between the particles, Dr. Herbert Chatley suggests the following formula[40]:

$$V_e = \frac{0.02}{D} \qquad (47)$$

in which D is a diameter of the particle, in centimeters; and V_e is a velocity in centimeters per second, which will just produce the necessary friction force to rupture the molecular bonds; that is, to erode the fine material.

Although the application of Equation (47) is perhaps limited, it offers a concrete idea as to the influence of the fine particles on scouring. This is illustrated by Table 11 in which the velocity, V_e, is computed by Equation (47). There is no definite limit to the size of colloidal particles, but it is generally assumed that particles with a diameter smaller than 0.001 mm (1 micron) are colloidal. For such particles, $V_e = 6.56$ ft per sec. On the other hand, the hydraulic value of a particle represents the gravity action in the water and is equal practically to zero for colloids. The relative values of cohesion and gravity for other sizes of particles are given in Table 11.

TABLE 11.—ERODING VELOCITY AND HYDRAULIC VALUE

Class of particle	Diameter D, in millimeters	Eroding Velocity, V_e: In centimeters per second	Eroding Velocity, V_e: In feet per second	Hydraulic value, ω, in feet per second
Fine gravel to coarse sand	1.00	0.2	0.0066	0.3280
Fine sand to very fine sand	0.10	2.0	0.0656	0.0262
Very fine sand to silt	0.05	4.0	0.1312	0.0095
Silt to clay	0.005	40.0	1.3120	0.0001
Colloids	0.001	200.0	6.5600	0.00004
Clay	0.0001	.....	......	......

There is a large portion of the material in sediment and soils of the Imperial Valley Project, which passed a No. 300 sieve (D = about 0.0043

[39] *Transactions*, Am. Soc. C. E., Vol. 100 (1935), p. 857, Table 8.

[40] "Problems in the Theory of River Engineering", by Herbert Chatley, Lond., 1929, p. 17.

mm), that approximately separates silt and clay. The presence of colloids in this class makes the sedimentary material rather coherent and considerably affects the scouring resistance, especially of the banks of the Imperial Valley canals. The steep banks of the channels in many localities, including the Imperial Valley, are explained by the cohesion forces of the very fine particles.

Silting and Scouring Limits.—There are two critical conditions or limits in a non-silting non-scouring channel. The low limit—the non-silting section—is the point at which the deposition of material stops. The upper limit—the non-scouring section—is the point at which the scouring begins. It has been established that even the sediment recently deposited would not scour again until the velocity was increased to the larger value of the silting velocity. The extent between these two limits might be considerable.

TABLE 12.—GENERAL HYDRAULIC FACTORS; EAST HIGHLINE CANAL AT "B" HEADING; IMPERIAL VALLEY, CALIFORNIA (S = 0.000322)

Item No.	Date	Hydraulic radius, R, in feet	Velocity, V, in feet per second	Discharge, Q, in cubic feet per second	Kinetic factor, K	Coefficient of hydraulic similitude, Y	Hydraulic number, N	Depth, d, in feet	Width, B, in feet	Width-depth ratio, r.
1	1- 7-1928	3.42	2.16	405.6	0.042	10.1	49.3	4.0	46.5	11.6
2	12-25-1928	3.35	3.25	593.5	0.098	15.8	50.5	3.9	46.6	11.9
3*	11-19-1928	3.68	3.47	705.1	0.102	14.1	44.4	4.4	46.8	10.7
4*	1-28-1928	3.94	3.62	801.2	0.103	13.1	40.8	4.7	46.6	9.9
5*	10-10-1928	4.17	3.83	900.4	0.109	12.4	37.4	5.1	46.1	9.1
6*	10-27-1928	4.45	3.90	1 000.9	0.106	11.3	34.8	5.50	46.6	8.5
7*	10-21-1928	4.61	4.10	1 100.6	0.113	11.2	33.4	5.7	47.0	8.2
8	1-28-1928	4.66	4.45	1 201.6	0.132	11.8	32.6	5.8	46.4	8.0
9*	9- 4-1928	5.03	4.35	1 301.6	0.117	10.2	29.9	6.4	46.3	7.3
10	5-23-1928	5.00	4.69	1 399.7	0.137	11.2	30.2	6.3	47.0	7.4
11	7-28-1928	5.60	4.58	1 598.5	0.116	9.1	26.7	7.3	47.4	6.5

* Sections considered non-silting and non-scouring by William T. Collings, Jr., M. Am. Soc. C. E.

As a demonstrative example in Table 12, Mr. Collings' data for the East Highline Canal, at "B" Heading, are used once more. Among the eleven observations, six are considered by Mr. Collings as non-silting non-scouring cross-sections. It appears that the first of these sections (Item No. 3, Table 12) is for the conditions when the silting stops, and the last observation (Item No. 9) is the upper limit related to the beginning of scouring. Since the kinetic factor in all the six observations is approximately the same, indicating that the conditions at the bottom do not vary much, attention should be directed to the stability of the banks. The widths of cross-sections shown in Table 12 and calculated under the assumption that the channels are of rectangular form, remain constant, their average value being 46.5 ft ± 1 per cent. This fact proves the stability of the banks, although the velocity gradually becomes greater with the diminishing width-bed ratio, both conditions tending to intensify the scouring action of flowing water on the banks. The degree of sediment concentration in flow at the time the measurements were taken, and the character of material composing the banks, are not known to the writer However, an examination of Table 12 as well as con-

sideration of other data, suggests that the value of the hydraulic number is a controlling factor of the stability of banks.

The hydraulic number becomes smaller in all the eleven cross-sections, ranging from 44.4 to 29.9 in the six stable sections. The other three stable canals indicated in Table 10 have the same diminishing tendency in regard to the hydraulic number. The study of canals in British India leads the writer to a suggestion[41] that their average hydraulic number is about 40. The average value of N for the four Imperial Valley canals, cited previously, is 43.5 and exactly 40, if the Alamo Canal is omitted. However, the range of these values is great, being approximately 65 to 30. The variation in flow when it extended beyond the non-silting and non-scouring limits, the large velocities, the difference in resistance of bed and banks, and the influence of original designs and methods of maintenance are probable reasons for such a range in the value of N. The hydraulic number in the two unstable canals of Table 10 has also the same diminishing tendency, approaching the value of 20. A reference has already been made to several erodible drainage canals[42] in which the hydraulic number varies from 15 to 22.50. The tendency for some of the foregoing values to coincide suggests that a hydraulic number equal to about 20 is the average upper limit for stable channels in "earth."

Variations in Discharge.—Examination of Table 12, and the previous remarks in regard to the non-silting and non-scouring limits for a cross-section, seem to indicate a method for the design of a canal with a varied flow. Two sections are drawn on Fig. 11. One section in solid lines is the actual section of Item No. 9, Table 12; that is, the upper limit mentioned previously. Another section in dotted lines is the non-silting section for the same discharge of 1 301.6 cu ft per sec, but the kinetic factor and hydraulic number are of Item No. 3 (Table 12); that is, the low limit. These two sections, or any intermediate section, might be taken for a prototype to compute and design the section corresponding to the given discharge. Each of these sections will be stable. Incidentally, that is one of the principal reasons for the large variation of width-depth ratio in the canals of the Imperial Valley.

If the discharge is variable, the upper limit may be used for the maximum discharge, or the low limit for the minimum discharge. The spread between these two limits will determine the possible variation in the discharges. If the spread could not include the given variation in discharge, it would be impossible to design a stable channel. There remains the possibility of devising a more or less "self-maintaining" channel with an unvarying tendency to silt or scour, the latter usually being preferable for irrigation canals.

All-American Canal.—Among the various factors introduced and discussed, the writer has intended mainly to bring out the importance of two factors, the kinetic factor and the hydraulic number, which, in his opinion, may be used as criteria for the stability of a channel in erodible material. Its bed stability is controlled by the kinetic factor, whereas the hydraulic number is related to the scouring resistance of banks.

[41] *Transactions*, Am. Soc. C. E., Vol. 100 (1935), p. 853, Table 5.

[42] *Loc. cit.*, p. 857, Table 8.

The author presents a study based on the design of the All-American Canal, and it would be of interest to see how the writer's discussion might be applied to the cross-sections adopted for this canal. The hydraulic elements[43] are, as follows, for its largest section: $Q = 15\,155$ cu ft per sec; $A = 4\,041.3$ sq ft; $V = 3.75$ ft per sec; $R = 16.63$ ft; $d = 20.61$ ft; and, by computation, $K = 0.02626$ and $N = 20.33$.

Assuming a rectangular cross-section, the foregoing values of K and N yield values of $d = 19.9$ ft and $r = 10.2$. A point corresponding to this depth and width-depth ratio is shown in Fig. 9 (see Point A, the All-American Canal) near the curves for the Egyptian canals, indicating that conditions in such channels are more or less similar to those assumed for the design of the All-American Canal.

The foregoing kinetic factor in the All-American Canal is not comparable to its prevailing value in the Imperial Valley on account of the proposed desilting works. A proper procedure for checking its value by Equation (35) would be to determine the "sediment equivalent" from a mechanical analysis of silt that is assumed to enter the canal; but such an analysis has not been made as far as the writer is aware. The concentration of the sediment near the bed should be known. Assuming that this concentration would be 10% higher than the average and that the coefficient, α, $= 1$, the mean hydraulic value computed from Equation (35) is $\omega_m = 0.01$ sec-ft. This hydraulic value corresponds approximately to the 280-mesh sieve, but that gives only a general indication of predomination of the finest particles in the sediment. Further study is required to derive a quantitative expression of the relation between the hydraulic number and the resistance of the material composing the banks. However, it is of considerable interest to note that the hydraulic number in the All-American Canal ($N = 20.33$) coincides with the value of 20, which has been suggested in the discussion as the average hydraulic number for canals in "earth."

Assuming that the proposed cross-section of the All-American Canal would tend to adopt a shape corresponding to the kinetic factor and hydraulic number as calculated previously, its depth should be less, whereas the width should be greater, than in the proposed cross-section. In other words, there might be a slight tendency to scour especially along the low parts of banks. Assuming, furthermore, that the hydraulic number is correct, this tendency to scour would be slight and would occur only at the time of full flow in the canal. The smaller the flow and, consequently, the cross-section, the greater will be the tendency for the silting to develop. These conditions, as the writer understands them, are desirable for the Imperial Valley because the collection of the settling sediment in the main canal is preferable to its conveyance to the distribution system.

W. M. GRIFFITH,[44] ESQ. (by letter).[44a]—Because it gives no new data, or mathematical reasoning to support or refute theories already advanced on this interesting subject, this paper is disappointing.

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[43] *Engineering News-Record*, October 17, 1935, p. 539.

[44] Cambridge, England.

[44a] Received by the Secretary May 19, 1936.

Fig. 3 was presented to show that high velocities extend closer toward the sides in the narrow, deep cross-sections than in the broad, shallow ones. This fact in itself would not appear to offer a satisfactory explanation to the well-known phenomenon that channels carrying heavy silt loads tend to adopt broad shallow cross-sections because, for any given discharge and surface slope, channels will be found to carry heavier silt loads if they are given broad shallow cross-sections than if they are given deep narrow ones, even if the sides are protected against erosion by a non-erodible lining.

The author quotes Mr. Kennedy's classic formulas (1)[44b] of relationship between velocities and depth of non-silting channels, and a large number of subsequent formulas, based on the same form of analysis of existing canal systems, in many parts of the world, which have values in many cases different from those of the channels cited by Mr. Kennedy. All these equations are in the form, $V_0 = C\, d^n$ (see Equation (1)) in which V_0 is the velocity that gives stability, and d is the depth of the channel section.

He also quotes Mr. Lacey's formula (18) (see Equation (7)), which may also be written in the form,

$$V_0 = C\, R^n \qquad (47)$$

and the essential difference between it and those based on Kennedy's studies, is that it claims that the stable velocity is a function of the hydraulic mean depth of the channel, and not of the depth of the channel. It is a more logical formula, in that the principal function, R, is governed by the general shape of the section, whereas d, the depth, is not. In an irregular cross-section it might be difficult to decide what value to take for d; that is, the average depth of the bed, or the maximum depth of the bed.

In consequence, it is not surprising that the values of the indices in these formulas based on Kennedy's studies differ. Kennedy did not claim that his formula was a basic law applicable to all channel sections; it referred to the Lower Bari Doab channel sections, which had approximately the same type of cross-section; namely, side slopes averaging 1 on 0.5 and presumably a bed level across the cross-section.

It has been found that none of these formulas has a universal application, and opinion has been expressed that it is not possible to write formulas for silt transportation and the hydraulic conditions governing a stable section, that have more than a local application.

In a paper entitled "A Theory of Silt and Scour",[45] the writer outlined a theory of silt transportations and the hydraulic conditions governing stable channel sections, which he advanced as the result of experience both in river training work and canal maintenance. This theory was based on the assumption that the power to transport silt depended only on the relative strength of the vertical or resultant vertical eddies, and was independent of that of horizontal eddies or of resultant horizontal eddies.

The main observation on which the theory was based was as follows: At any point in a flowing cross-section heavily charged with silt, if the velocity

[44b] For reference to figures in parentheses, see "Bibliography", *Proceedings*, Am. Soc. C. E., November, 1935, p. 1325.

[45] *Minutes of Proceedings*, Inst. C. E., Vol. 223 (1927) pp. 243 to 251.

is reduced artificially the depth is found to reduce automatically due to silting, and if the velocity at a point is increased, the depth is found to increase due to scour.

In consequence, the writer was led to the conclusion that Kennedy's relationship between depths and velocities (Equation (1)) was in reality a basic law of silt transportation, not for channel sections considered as a whole, but giving the connection between the silt charge transported and the hydraulic conditions necessary for its transportation at a point.

It follows from this general assumption (that in a flowing section, equilibrium at all points requires that Equation (1) shall be satisfied, and from Chezy's basic law, that if the value of the exponent, n, is equal to 0.5, the section can be in equilibrium whatever its shape. On the other hand if the value of n is less than 0.5 the section must be deep and narrow, for equilibrium, and *vice versa*. As natural sections carrying heavy silt charges are invariably broad and shallow it follows that the value of n must be greater than 0.5.

In the writer's paper[45] referred to, the value of n was assumed to be 0.64. As the examples dealt with were for river beds of shingle and boulders this value was consistent. For sand, however, a lower value of n is indicated. Arguing mathematically from this basic assumption it follows, considering any cross-section, that for equilibrium as a whole, by integrating Equation (1) across the section the condition for equilibrium is that, necessarily,

$$V_m = C\, d^n_m \qquad (48)$$

in which V_m is the mean velocity of the entire cross-section and d_m is the mean depth. In Equation (48), as in Lacey's formulas, the main function governing the critical velocity (that is, the mean depth) is controlled by the shape of the cross-section. On the other hand, while from this theory it follows mathematically that the bed of the cross-section for equilibrium must be level across the section, Lacey postulates a curved bed which, in its true form, he claims to be semi-elliptical.

The difference between Equation (47) and Equation (48) is fundamental, because the sum of the eddies created by the friction of the flow is a function of R, the hydraulic mean depth, whereas the sum of the vertical eddies only, or their vertical components, is a function of d_m, the mean depth.

In other words Lacey postulates a semi-elliptical stable cross-section and a law of silt transportation depending on all eddies created by the flowing section, and the writer, a cross-section having a horizontal bed and a law of silt transportation depending on the vertical eddies or the vertical components of the eddies only.

The writer advanced his theory as a general theory of silt transportation, but, in reality, it relates to loose granular material, such as boulders, shingle, and sand, graded down to a size of the order of $\frac{1}{300}$ in. in diameter, but it does not relate to the transportation of a silt charge containing only such light material as is held in suspension and transportable under the action of all eddies.

In practise, normally, a silt charge will contain silt of both classes—heavy and light—but such a case is governed by the law that controls the heavier class only, because if the heavier class is transported, the lighter material will remain in suspension automatically. There are cases, however, in which the silt charge consists of very light material only (such as clay, organic matter, and sand of an extreme order of fineness), and in such cases the theory advanced by the writer[45] does not apply. The two different sets of conditions, perhaps, can best be illustrated by considering a canal or a canal system carrying at its head a heavy silt charge consisting of both classes of material. The sections of the channel or channels near the head will have horizontal beds and the beds will be of pure sand, the great majority of the grains being more than $\frac{1}{300}$ in. in diameter. If the system is a long one, the silt of a heavier order will leave it through the distributing outlets which being situated near the bed level draw off the heavier silt particles. At the tails of this system, then, only light material will be left to be transported and the type of cross-section, if left to form its natural section, will change from a level bed to a curved bed, possibly elliptical in shape as claimed by Lacey.

Now, in such a system the question of designing a non-silting section correctly is usually of importance only in the upper reaches carrying the silt of a heavy order, because the rate of settlement of silt of the lighter order is very small and the material settled on the perimeter at the tail of the sye-tem is not pure sand; it has binding properties and is useful for forming the channel banks.

If, therefore, as the writer suggests, there are, in fact, two different conditions of silt transportation, one depending on the ratio, $\frac{V}{d^n{}_m}$, and the other on $\frac{V}{R^n}$ or $\frac{V}{R^2}$, it would explain why relationships argued mathematically from the latter formulas would not be applicable to the conditions pertaining in the Imperial Valley Canals, which carry silt of the heavy order.

In Fig. 2 the author has plotted on logarithmic paper the mean depths against bed widths of a number of channels, including those considered to be in regime on different canal systems, apparently, with a view to determining whether any general relationship between bed width and mean depths can be established for non-silting channels. The result shows very clearly that no such relationship exists.

If the general formula, Equation (48), is correct, if B = width at water surface, and Q is the discharge, then as $V_m = \frac{Q}{B\,d_m}$: From Equation (48), $\frac{Q}{B\,d_m} = C\,d^n{}_m$. Therefore,

$$B = \frac{Q}{C\,d^{1+n}{}_m} \quad \text{.......................} (49)$$

Equation (49) would indicate that it is not possible to frame a general law of relationship between width and depth irrespective of the discharge or the silt load carried.

Table 1 gives values of C and n (in a general formula of the Kennedy type —Equation (1)) which are found to give values for non-silting channels in different canal systems in different countries.

The differences in C are understandable as the silt load will vary with the conditions of the different rivers feeding the canal systems. The big differences in value of n shown in Table 1, however, are illogical if all beds are of graded sand. In model experiments normal differences in the size of sand are found not to affect the coefficient of roughness. In the case of boulders, or boulders and shingle, where the vertical eddies are materially affected by the obstruction of particles of an entirely different order of size, a different value of n might be expected, but it is not reasonable to expect big differences in silt containing graded sand.

If, however, as the writer claims, Equation (1) is inapplicable to the channel section considered as a whole, and should be of the form of Equation (48), then these differences are explainable.

For example on the assumption of Kennedy's law (Equation (1)), the data of the channels cited by Kennedy are found to conform to the equation,

$$V = 0.84\ d^{0.64} \qquad (50)$$

(Table 1, Curve No. 17) and the Godavari Western Delta, Madras (Table 1, Curve No. 9),

$$V_0 = 0.67\ d^{0.55} \qquad (51)$$

This gives a wide difference in values of n. If, however, both sets of data are plotted in terms, not of d, the depth, but of d_m, the mean depth, Kennedy's data are found to conform to the law,

$$V_m = 0.97\ d^{0.57}_m \qquad (52)$$

and the Godavari Western Delta, Madras, to

$$V_m = 0.69\ d^{0.57}_m \qquad (53)$$

and this difference in the values of n has disappeared.[46]

In his research the author is presumably seeking for the most efficient type of cross-section, namely, that relationship of width and depth which will give the channel its greatest silt-carrying capacity for any given discharge and surface slope.

With Equation (48), the maximum silt-carrying capacity is attained when $\frac{V}{d^n_m}$ is a maximum. If B = the width at the water surface,

$$V = \frac{Q}{B\ d_m} \qquad (54)$$

[46] "Stable Channels in Alluvium", *Minutes of Proceedings*, Inst. C. E., Vol. 229 (1930), p. 321.

and, therefore,

$$\frac{V}{d^n{}_m} = \frac{Q}{B\, d^{1+n}{}_m} \quad \text{.......................} (55)$$

Consequently, for any given discharge, the maximum silt-carrying capacity is attained for the cross-section in which $B \times d^{1+n}{}_m$ is a minimum.

Accepting, for sand, the value of n obtained from the data used by both Kennedy and for the Godavari Western Delta, Madras (that is, $n = 0.57$) then the maximum silt is carried when $\frac{V}{d^{0.57}{}_m}$ has a maximum value, or for any fixed discharge when $B \times d^{1.57}{}_m$ has a minimum value. If Kennedy's equation is correct the maximum silt-carrying capacity is attained when $\frac{V}{d^{0.64}}$ has a maximum value.

To compare these two different equations of silt-carrying capacity or efficiency, it is necessary to take some particular case as, for example, a channel having a discharge of 100 cu ft per sec, a surface slope of $\frac{1}{4\,000}$ and side slopes of 1 on 0.5, carrying a heavy silt load.

TABLE 13.—HYDRAULIC DATA PERTAINING TO DIFFERENT CHANNEL SECTIONS

(Discharge = 100 cu ft per sec; surface slope = $\frac{1}{4\,000}$; side slopes, 1 on 0.5; and discharge is calculated by Kutter's formula for $n = 0.0225$)

Item No.	Bottom width, B_B, in feet	Depth of channel, d, in feet	Area of channel section, A, in square feet	Velocity, V, in feet per second	Width, B, of channel at the water surface, in feet	d_m = mean depth of channel section, in feet	Values of $d^{0.57}_m$	Silt-carrying capacity, $\frac{V}{d^{0.57}m}$	Values of $d^{0.64}$	Silt-carrying capacity, $\frac{V}{d^{0.64}}$
1...	54.0	1.5	82.12	1.22	55.5	1.48	1.25	0.978	1.298	0.942
2...	47.0	1.6	76.98	1.30	48.6	1.64	1.305	0.999	1.343	0.968
3...	43.0	1.7	74.45	1.34	44.7	1.66	1.335	1.003	1.387	0.97
4...	33.0	2.0	68.0	1.47	35.0	1.94	1.46	1.008	1.56	0.944
5...	23.0	2.5	60.62	1.65	25.5	2.42	1.652	1.00	1.80	0.920
6...	17.0	3.0	55.5	1.81	20.0	2.725	1.77	0.99	2.045	0.886
7...	13.5	3.5	53.4	1.88	17.0	3.14	1.92	0.98	2.23	0.842
8...	10.9	4.0	51.6	1.94	14.9	3.46	2.03	0.96	2.43	0.80
9...	8.8	4.5	49.6	2.01	13.3	3.77	2.13	0.945	2.60	0.772
10...	7.4	5.0	49.5	2.02	12.4	4.00	2.20	0.92	2.80	0.720
11...	5.9	5.7	49.87	2.01	11.6	4.30	2.30	0.875	2.98	0.675

In Table 13 are given the calculated hydraulic data of eleven different designs of channel sections giving this discharge, ranging from a 1.5-ft depth, with 54-ft bed width, to 5.7-ft depth and 5.9-ft bed width. Values of $\frac{V}{d^{0.57}{}_m}$ and $\frac{V}{d^{0.64}}$ are also given.

Fig. 12, for this case, shows that if the silt-carrying capacity is correctly expressed in terms of $\frac{V}{d^{0.64}}$, as Kennedy suggests, the silt-carrying efficiency

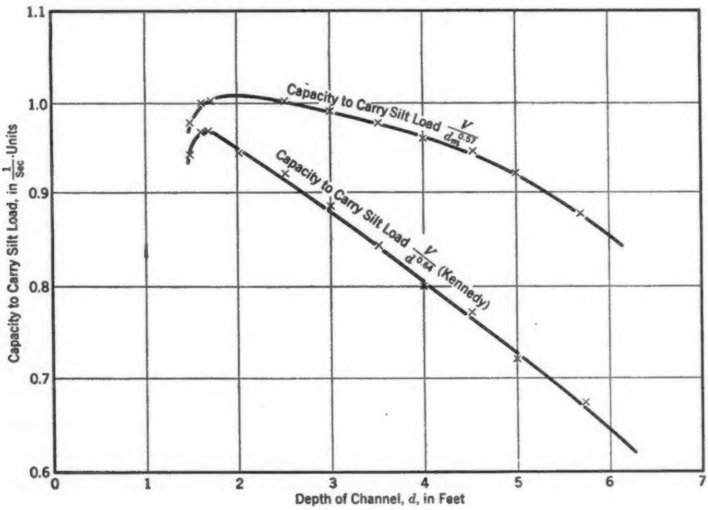

FIG. 12.

will increase uniformly and rapidly from a channel of 6-ft depth to 1.7-ft depth, at which time the maximum efficiency is reached, and a sudden change occurs in the efficiency curve.

If, on the other hand, the correct expression is $\frac{V}{d^{0.57}_m}$, the silt-carrying efficiency at first increases rapidly as depths are reduced from 6 ft downward, but the curve flattens and shows little difference in silt-carrying capacity between a channel of 3-ft depth and one of 2.5-ft depth. The maximum efficiency appears to lie with a channel 2 ft deep.

The writer's experience with channels of this discharge and slope indicates that in practice the expression, $\frac{V}{d^{0.57}_m}$, gives, more correctly, the value of silt-carrying efficiency of the various types of section.

AMERICAN SOCIETY OF CIVIL ENGINEERS

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DISCUSSIONS

LATERAL PILE-LOADING TESTS

Discussion

BY MESSRS. LAZARUS WHITE, AND Y. L. CHANG

LAZARUS WHITE,[14] M. AM. SOC. C. E. (by letter).[14a]—The experiments made by Mr. Feagin and his associates are familiar to the writer and the paper describing them constitutes a valuable contribution in a field in which reliable information is scanty. The conditions under which the experiments were conducted are carefully described and the author's conclusions are properly limited to the soil conditions and piling arrangements under which the tests were made.

It will interest most engineers to note that the lateral resistance of the piles tested was less than the ratios ordinarily assumed in such designs. That piles about 30 ft long—of white oak and concrete, driven into firm sand, and capable of sustaining a vertical load of, say, 30 tons, with their ends firmly fixed into a solid concrete block—should have a resistance of 4 to $4\frac{1}{2}$ tons for a lateral movement of $\frac{1}{4}$ in., is rather startling. It is startling, furthermore, to discover that well made concrete piles have a lateral resistance of only 1 ton or 2 tons more. For a lateral movement of $\frac{1}{2}$ in. the indicated lateral load is only 7 tons. That the values are really low is borne out by observations on the movements of walls founded on wooden piles driven in sand (see Fig. 26). These walls moved several inches.

Tests made on the holding-down power of wooden piles indicate a high value in most cases. A vertical load of 20 to 75 tons is required to pull one pile out of a concrete block, so that it is probably correct to assume fixed-end conditions.

The writer wishes to call attention to the danger of extending observations or calculations made on single piles or on a small group, to a large group, such as is ordinarily done The piles of a large group may grip enough soil to act as a diaphram much as one may retain a bank with longitudinal or

NOTE.—The paper by Lawrence B. Feagin, Assoc. M. Am. Soc. C. E., was published in November, 1935, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: November, 1935, by A. E. Cummings, Assoc. M. Am. Soc. C. E.; January, 1936, by Messrs. J. C. Meem, and T. Kennard Thomson; and February, 1936, by August F. Niederhoff, Jun. Am. Soc. C. E.

[14] Pres., Spencer, White & Prentis, Inc., New York, N. Y.

[14a] Received by the Secretary March 18, 1936.

vertical sheeting with wide open joints. In this case the resistance against lateral movement is that of an approximately vertical plane of earth bounding the group, much the same as the resistance of a line of very stiff steel sheeting.

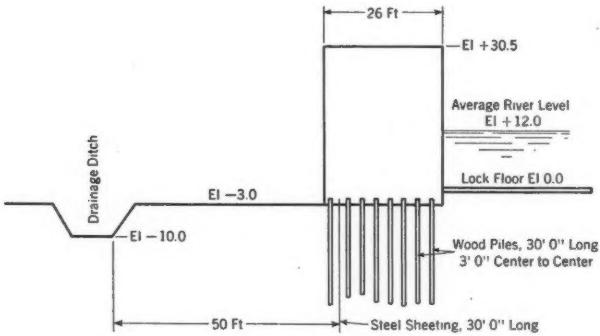

FIG. 26.

The writer has observed large movements of groups of piles serving as foundations to concrete walls. When the lateral resistance of the earth surrounding them has diminished due to near-by excavations, the walls move as a unit. These walls were also supported on lines of steel sheeting serving as a cut-off walls. Much of this dimunition of resistance, in the writer's opinion, was due to the lateral flow of water in the adjacent excavation, rendering the sand surrounding the piles partly quick. It is apparent, therefore, that under field conditions, the lateral resistance of piles may be highly variable, and, at times, it may be only a few tons per pile. The lesson is that under the conditions described it is wise to supplement the lateral resistance of vertical piles with other, more positive, means. Steel sheeting below the walls does not provide this resistance, as, in the cases quoted, the tops of the steel sheeting utilized as cut-offs were incorporated in the wall. The more positive means may be struts or ties or battered piles, advantageously placed.

Y. L. CHANG,[15] ESQ. (by letter).[15a]—These valuable tests on the lateral deflection of foundation piles give rise to much useful information on this subject not hitherto available. The need of some definite information and a method of analysis is certainly urgent, owing to the rapidly increasing use of piles and sheet-piling in foundation work. Mr. Feagin has taken the opportunity of conducting tests on piles driven in river sand, with a view to determining the resistance of such piles to movement under lateral loads caused by back-fill on a lock or by water pressure on a dam. The Huai River Commission in China is at present (1936) constructing three locks on the Grand Canal to improve navigation, and, at the same time, to maintain a definite

[15] Scholar of Board of Trustees, Nanking, China; Research Student, Victoria Univ., Manchester, England.

[15a] Received by the Secretary March 20, 1936.

slope to facilitate local irrigation. A general knowledge of the lateral deflection of piles, such as that presented by the author, will certainly be of great help in such a case.

The problem is equally interesting in the case of a sheet-pile; and, since it is usually designed with a comparatively low section modulus, an investigation of its movement under lateral load is of special importance.

The author has mentioned that an exact mathematical analysis to verify the experimental data is not at all easy, owing to the fact that so many variables are involved and that conditions vary widely for different localities. An attempt at such an analysis, with the aid of model tests of the pile, would provide a close guide to apply in a given case. When small elastic rods are embedded in granular material and subjected to lateral pressure they will bend in the form of a reverse curve, and the bottom of the rod will move slightly in a direction opposite to that of the displacement at the top. At a certain intermediate depth there is rotation, but no displacement.

An approximate analysis is suggested herein, involving the assumption that the "elastic modulus of soil", E_s, is a constant, and it follows that the foregoing phenomena can be demonstrated quite rationally. By assuming an arbitrary value for E_s (the true value of which should be determined at the site locally), a curve can be plotted to agree quite satisfactorily with Mr. Feagin's experimental data.

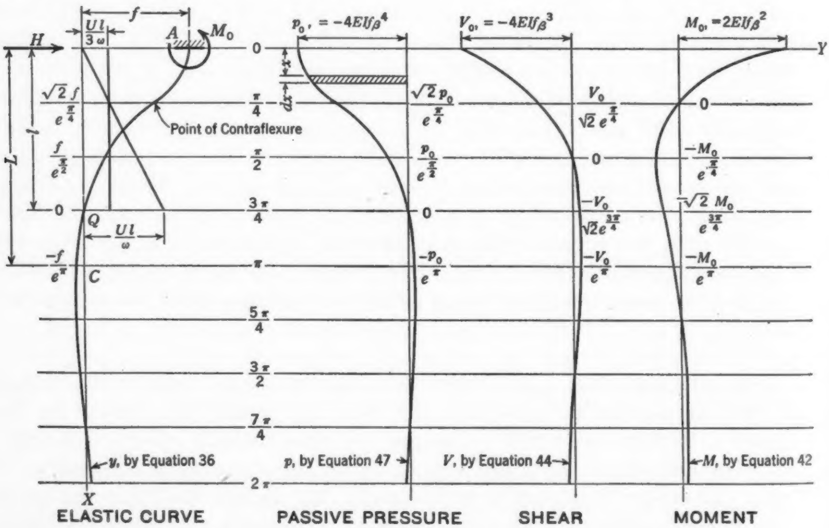

FIG. 27.

The assumptions upon which the analysis is based are as follows:

(1) The upper end of the pile is embedded deeply enough in the monolithic foundation to be fixed against rotation;

(2) The pile is infinitely long (as will be shown subsequently, the lateral movement of a pile at depths greater than l, (Fig. 27) is so small that any pile of sufficient length may be assumed to be infinitely long);

(3) The elastic modulus of soil, E_s, is constant throughout the depth; and,

(4) The passive pressure, p, on a pile is proportional to its displacement; that is, $p = - E_s\, y$, in which y is the lateral deflection of the elastic curve.

Regarding Assumption (3), moreover, the soil itself will be compacted, more or less, to the same degree and uniformity, due to the overlapping of the pressure cones that are created when all the piles are driven to their proper depths. Since most foundations are designed on the assumption that the supporting power is distributed properly both on the piles and on the bearing area of the earth foundation, it follows that the earth must be quite compact even at the surface. Therefore, it is not necessary to assume that the "elastic modulus of soil" is a straight line, varying with the depth. As a matter of fact, its value depends on the properties of the soil at different strata and on the intensity and the distribution of the vertical loads. If it is assumed, tentatively, that this constant modulus is one-third of what the variable modulus would be at the depth, l (Fig. 27), a simple substitution can be made. The reason for making E_s less than the mean value of a triangular variation may be justified by considering that the upper part of a pile, being subjected to much more pronounced lateral movement, is surrounded by earth of comparatively low elasticity. It will be shown subsequently that l is proportional to $\frac{1}{\sqrt[4]{E_s}}$, so that if an error of 10% is made in the value of E_s, there would be only an error of 2% in the value of l.

Referring to Assumption (4), the ordinary differential equation[16] is,

$$EI\,\frac{d^4 y}{dx^4} = p = - E_s\, y \ldots\ldots\ldots\ldots\ldots\ldots (32)$$

in which E = the modulus of elasticity of the pile material.

Let, $\beta = \sqrt[4]{\frac{E_s}{4\,EI}}$ and, the general solution is,

$$y = e^{\beta x}\,(A \cos \beta\, x + B \sin \beta\, x) + e^{-\beta x}\,(C \cos \beta\, x + D \sin \beta\, x)$$

in which e is the base of Naperian logarithms, and A, B, C, and D are arbitrary constants. As $e^{\beta x}$ increases with x, Constants A and B must be equal to zero to satisfy the condition, $y = 0$ at $x = \infty$; therefore,

$$y = e^{-\beta x}\,(C \cos \beta\, x + D \sin \beta\, x) \ldots\ldots\ldots\ldots\ldots (33)$$

As the top is fixed against rotation, $\frac{dy}{dx} = 0$ when $x = 0$; or,

$$\frac{dy}{dx} = - \beta\, e^{-\beta x}\,[(C - D) \cos \beta\, x + (C + D) \sin \beta\, x] = 0 \ldots (34)$$

since $C - D = 0$, and, therefore, Equation (33) becomes,

$$y = C\, e^{-\beta x}\,(\cos \beta\, x + \sin \beta\, x) \ldots\ldots\ldots\ldots\ldots (35)$$

[16] "Strength of Materials", by S. Timoshenko, Pt. II, p. 402, Equation (1).

At $x = 0$, $y = f$, from which $C = f$; and,

$$y = f\, e^{-\beta x} (\cos \beta\, x + \sin \beta\, x) \quad \text{.......} (36)$$

and,

$$\frac{dy}{dx} = -\, 2\, \beta\, f\, e^{-\beta x} \sin \beta\, x \quad \text{.......} (37)$$

The point of zero deflection (see Point Q, Fig. 27) can be found by making $y = 0$ and $x = l$ in Equation (36), from which,

$$e^{-\beta l} (\cos \beta\, l + \sin \beta\, l) = 0 \quad \text{.......} (38)$$

One solution of Equation (38) is given by $l = \infty$; or, $\cos \beta\, l + \sin \beta\, l = 0$; $\tan \beta\, l = -\, 1$; and,

$$\beta\, l = \left(n - \frac{1}{4}\right) \pi \quad \text{.......} (39)$$

in which $n = 1, 2, 3$, etc. The foregoing analysis demonstrates that the elastic curve for an infinitely long pile should be a harmonic wave, damping away rapidly after each successive wave length, $2\, l$. For the least value, $\beta\, l = \frac{3\, \pi}{4}$, and,

$$l = \frac{3\, \pi}{4\, \beta} = \frac{3}{4}\, \pi \sqrt[4]{\frac{4\, EI}{E_s}} \quad \text{.......} (40)$$

For the point of zero slope, $\frac{dy}{dx} = 0$, and $x = L$ in Equation (37); therefore, $e^{-\beta L} \sin \beta\, L = 0$. In one solution, $L = \infty$; or, $\sin \beta\, L = 0$; and, $\beta\, L = n\, \pi$. The least value is $\beta\, L = \pi$, and,

$$L = \frac{\pi}{\beta} = \pi \sqrt[4]{\frac{4\, EI}{E_s}} \quad \text{.......} (41)$$

By proportion, $l = \frac{3}{4} L$. It is to be noted that for $n = 2$, L is very nearly equal to l, showing that the pile is practically vertical and rigid at a depth of one wave length. At the depth, L (Fig. 27), the lateral movement in the opposite direction is a maximum. To find this deflection, substitute $\beta\, x = \pi$ into Equation (36), and $y = -\frac{f}{e^{\pi}}$, which amounts to 4.3% of the top deflection in the opposite direction.

For the point of maximum slope, differentiate Equation (37) with respect to x; thus: $\sin \beta\, x = \cos \beta\, x$; and $\beta\, x = \frac{\pi}{4}$. To compare the foregoing with Equations (40) and (41), $x = \frac{l}{3} = \frac{L}{4}$. To find an expression for

moment, Equation (37) is differentiated further:

$$\frac{d^2 y}{dx^2} = 2\ f\ \beta^2\ e^{-\beta x}\ (\sin \beta\ x - \cos \beta\ x);$$

and,

$$M = -\ 2\ E\ I\ f\ \beta^2\ e^{-\beta x}\ (\sin \beta\ x - \cos \beta\ x) \qquad (42)$$

At $x = 0$,

$$M_0 = 2\ E\ I\ f\ \beta^2 \qquad (43)$$

The point of zero moment, determined by $\sin \beta\ x = \cos \beta\ x$, occurs at the point of contraflexure. Differentiating again:

$$\frac{d^3 y}{dx^3} = 4\ f\ \beta^3\ e^{-\beta x}\ \cos \beta\ x \qquad (44)$$

At $x = 0$, the shear, V_0, equals:

$$V_0 = -\ 4\ E\ I\ f\ \beta^3 = -\ H \qquad (45)$$

in which H = the external force applied at the top of a pile, and, at $x = L$,

$$V_L = 4\ E\ I\ f\ \beta^3\ e^{-\pi} = -\ \frac{V_0}{e^{\pi}} \qquad (46)$$

which is only 4.3% of V_0.

For the point of zero shear, or for that of maximum moment, $\cos \beta\ x = 0$; $\beta\ x = \left(n - \frac{1}{2}\right)\pi$; and the first point occurs at a depth of $\frac{L}{2}$.

The passive pressure on the pile is:

$$p = -\ E_s\ y = -\ E_s\ f\ e^{-\beta x}\ (\cos \beta\ x + \sin \beta\ x) \qquad (47)$$

Considering a pile of infinite length, the total passive pressure on the pile is:

$$P = \int_0^\infty p\ dx = -\ E_s\ f \left[\int_0^\infty e^{-\beta x} \cos \beta\ x\ dx + \int_0^\infty e^{-\beta x} \sin \beta\ x\ dx \right]$$
$$= -\ 4\ E\ I\ f\ \beta^3 = -\ H \qquad (48)$$

If Equation (48) is integrated to the point of zero deflection,

$$P = \int_0^l p\ dx = 4\ E\ I\ f\ \beta^3\ e^{-\beta x} \cos \beta\ x \Big]_0^{\frac{3\pi}{4\beta}} = -\ 4\ E\ I\ f\ \beta^3 \left(1 + \frac{1}{\sqrt{2}\, e^{\frac{3}{4}\pi}}\right)$$
$$= -\ 1.067\ H \qquad (49)$$

The difference between the total passive pressure when assuming that the pile does not bend beyond the depth, l, is only 6.7% in excess of that on an infinitely long pile. This difference is due to the decreasing pressure intensity as the depth increases. The length of pile that is subject to passive pressure can also be determined by considering the minimum internal energy set up in a pile during elastic deformation, thus:

$$W = \frac{1}{2} \int_0^d p\ y\ dx = -\ \frac{1}{2} \int_0^d E_s\ f^2\ e^{-2\beta x}\ (\cos \beta\ x + \sin \beta\ x)^2\ dx \qquad (50)$$

in which the upper limit, d, denotes the depth, or, finally,

$$W = \frac{E_s f^2}{8 \beta} \left[e^{-2\beta d} \left(\cos 2 \beta d + \sin 2 \beta d + 2 \right) \right] \ldots\ldots\ldots (51)$$

As before,

$$\frac{\partial W}{\partial d} = - \frac{E_s f^2}{2} \left[e^{-2\beta d} \left(\sin 2 \beta d + 1 \right) \right] = 0 \ldots\ldots\ldots\ldots (52)$$

In one solution, $d = \infty$; or, $\sin 2 \beta d = - 1$; that is,

$$2 \beta d = \left(2 n - \frac{1}{2} \right) \pi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (53)$$

For $n = 1$, $2 \beta d = \frac{3}{2} \pi$, and,

$$d = \frac{3 \pi}{4 \beta} = l = \frac{3}{4} \pi \sqrt[4]{\frac{4 E I}{E_s}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (54)$$

which is also the expression for l in the case of zero deflection (see Equation (40)). Equation (54) shows that a similar expression for a definite bent length, l, can always be derived by considering either that the pile is rigid beyond this depth or that it is deformed in damping waves throughout the entire length.

Let E (the modulus of elasticity of the pile) = 270 469 440 lb per sq ft; I = 0.0503 ft^4; U = 133 lb per cu ft; and ω = 0.005. Then, by the foregoing assumption, $E_s = \frac{U l}{3 \omega} = \frac{133 l}{3 \times 0.005} = 8\,866 l$. These values, substituted in Equation (40) yield, l = 11.35 ft; or, E_s = 100 700 lb per sq ft. Then, by substituting, in the expression, $\beta = \sqrt[4]{\frac{E_s}{4 E I}}$, $\frac{1}{\beta} \cong \sqrt[4]{541}$; or $\frac{1}{\beta^3} = 112.2$. Equation (45) gives the relation between the horizontal load applied at the top of the pile and the lateral deflection of the top of the pile; that is, substituting the foregoing values and solving for f:

$$f = \frac{H}{4 E I \beta^3} = \frac{H}{20.2} \ldots (55)$$

in which f is in inches and H is in tons.

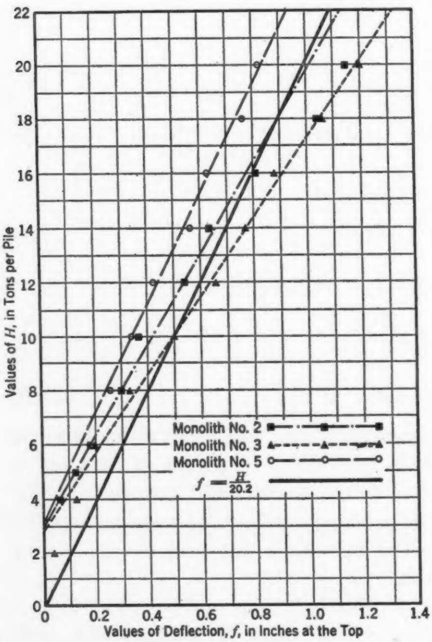

FIG. 28.

When Equation (55) is plotted (see Fig. 28) the curve has nearly the same slope as the experimental curves of M_2 and M_5 in Mr. Feagin's paper. In the author's "Conclusions", a value of 7 tons has been suggested as the maximum lateral load on the top of a pile if a lateral movement of not more than 0.5 in. is allowable. Adopting 0.5 in. as a practical limit, Fig. 28 gives 10 tons as the maximum lateral load. Reference to this curve shows that for deflections of less than 0.5 in., the experimental data for M_2, M_3, and M_5 follow lines of the same general slope as those defined by Equation (55); in fact, they are on the safe side.

The best empirical equation that will fit the field data is evidently of the form:

$$f = \frac{H}{a} - b \qquad (56)$$

in which a and b are constants.

In conclusion, the writer wishes to thank Professor A. H. Gibson and Mr. J. Allen, of the Victoria University of Manchester, England, and A. E. Cummings, Assoc. M. Am. Soc. C. E., for their helpful criticism of this discussion.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

WIND STRESSES IN REINFORCED CONCRETE ARCH BRIDGES

Discussion

BY MESSRS. FANG-YIN TSAI, AND PAUL ANDERSEN

FANG-YIN TSAI,[14] ASSOC. M. AM. SOC. C. E. (by letter).[14a]—The problem of wind stresses in reinforced concrete arch bridges is similar to that of bow girders, which have been treated previously by many authors[15], the only difference being that the former are fixed-end beams, curved in vertical plan, and subjected to horizontal loads, whereas the latter are fixed-end beams curved in horizontal plan, and subjected to vertical loads. However, the part of the paper under the heading, "Braced Arch Ribs", seems to be quite original. Although the solution of the problem has been indicated previously by some authors[16], the scant attention which the problem has received thus far, as well as the tendency to increase the span lengths of such bridges, makes the presentation of this paper very timely.

It is most unfortunate that the demonstrations are rather too brief and also not very clear. There is also quite a number of ambiguities which cause considerable confusion when one wishes to follow the demonstrations closely. For example, as stated by the author, Equations (3) and (4) are derived from Equation (2) by applying Castigliano's theorem[16a]. It may be noted that, for computing displacements, the application of Castigliano's theorem is advantageous only in cases where there is a load at the point and also in the direction of the desired displacement. Otherwise, an arbitrary load must be first introduced at the said point and in the said direction, and then the

NOTE.—The paper by A. A. Eremin, Assoc. M. Am. Soc. C. E., was published in December, 1935, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: April, 1936, by Leon Blog, Assoc. M. Am. Soc. C. E.

[14] Prof. of Structural Eng., Dept. of Civ. Eng., National Tsing Hua Univ., Peiping, China.

[14a] Received by the Secretary March 17, 1936.

[15] For a comprehensive bibliography on the subject, see *Journal*, Am. Concrete Inst., November, 1932, p. 158.

[16] See. for instance, discussion by Norman M. Steinman, *Journal*, Am. Concrete Inst., November, 1932, p. 153.

[16a] Correction to be made before paper is published in *Transactions*: The two lines preceding Equation (3) should read "Applying Castigliano's theorem, the vertical displacement at some point, N (Fig. 1), may be derived from Equation (2) as follows:"; and, the two lines following Equation (6) should read: "The partial differential coefficients of the unit moments, $m_1 = 1$ and $m_2 = 1$, are:"

load is set to zero after the partial differentiation[17]. Consequently, there is a considerably long step from Equation (2) to Equations (3) and (4). However, if the method of work with a dummy unit loading as developed by Maxwell, Mohr, and the late George Fillmore Swain[18], Past-President and Hon. M., Am. Soc. C. E., is used, Equations (3) and (4) can be written directly with the dummy unit moments, $m_1 = 1$ and $m_2 = 1$, applied, successively, at Point P. Therefore, it is much simpler and more straightforward than the application of Castigliano's theorem. Of course, the two methods differ only in conception and operation, and they give identical results.

Whereas the author states correctly that "* * * C_{1m} and C_{1t} are, respectively, the partial differential coefficients representing the moment and torsion produced by a unit moment, $m_1 = 1$, applied at Point P, * * *," he errs in stating (the sentence preceding Equation (7)) that C_{1m}, C_{1t}, etc., are the partial differential coefficients of the unit moments, $m_1 = 1$ and $m_2 = 1$. It must be also noted that the quantities, m_u and m_v, in Equation (2) are not the same as those in Equations (3) and (4). Whereas, those in Equation (2) are, respectively, the bending and torsion moments at Point N due to both the actual load (wind, m_1 or m_2, as the case may be) and the dummy moments, m_1 and m_2, applied at Point P in the horizontal and profile planes, those in Equations (3) and (4) are, respectively, the bending and torsion moments at Point N due to the actual load only. The values of m_u and m_v, due to the actual load, to be substituted in Equations (3) and (4), will be, respectively, those given by: (*a*) Equations (5) and (6) for deriving Equations (11) and (12) due to wind load[16a]; (*b*) $+ \cos\phi + M_1 \cos\phi$ and $- \sin\phi\, M_1 \sin\phi$, for deriving Equations (14) and (15), due to the symmetrical moments, $m_1 = 1$; and (*c*) $+ \sin\phi + M_2 \cos\phi$ and $+ \cos\phi - M_2 \sin\phi$, for deriving Equations (17) and (18) due to symmetrical moments, $m_2 = 1$. The foregoing distinction must be noted carefully in following the derivation of all the aforementioned equations in order to avoid confusion. Again, the complications in applying Castigliano's theorem to the problem are evident.

Following Equation (25) the author states that Equations (24) and (25) may be expressed by Maxwell's theorem. Apparently, he refers to the principle of superposition, which is not due to Maxwell, as pointed out elsewhere[19] by the writer.

Table 1 gives only the numerical results of the various terms of integration, without showing the detailed method of computing them. The writer hopes that, in his closing discussion, the author will show the detailed computations for all the terms in tabulated forms as A. J. S. Pippard[20], M. Am. Soc. C. E., did in solving a problem of bow girder. Such an example of tabulated computations will not only facilitate the application of the method, but will also help in understanding the various terms of integration. It will be certainly welcomed by all those who wish to use the method in practice.

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[17] See "Statically Indeterminate Stresses", by J. I. Parcel and G. A. Maney, Members Am. Soc. C. E., John Wiley, New York, First Edition, 1926, pp. 37–39.

[18] *Transactions*, Am. Soc. C. E., Vol. LXXXIII (1919–1920), p. 622.

[19] *Proceedings*, Am. Soc. C. E., May, 1935, p. 693.

[20] "Strain Energy Methods of Stress Analysis", by A. J. S. Pippard, Longmans, Green & Co., London, 1928, pp. 133–135.

PAUL ANDERSEN,[21] ASSOC. M. AM. SOC. C. E. (by letter).[21a]—This paper is a valuable contribution to the theory of arch design. It is incomplete to the extent that it makes no mention of stresses in barrel arches with spandrel walls. Although the bending stresses, due to wind load, in an earth-filled arch are of no consequence, torsional stresses at the springing line may become quite large.

Using the author's notation and applying the principles of statics, the maximum torsional moment, M_T, which occurs at the springing line can be determined from,

$$M_c \cos \alpha + M_T = \int w\, t\, n\, ds \qquad (34)$$

in which α = the angle between the horizontal line and the tangent to the earth axis at the springing line; and n = the distance from the arch element to this tangent.

Using the values from the illustrative example in Fig. 3, Equation (34) becomes, $92\,785 \times 0.715 + M_T = 99\,470$; and $M_T = 33\,130$ ft-lb.

This moment produces a maximum shearing stress, at the center of the wide side, of $\dfrac{33\,129 \times 12}{0.225 \times 54 \times 42^2} = 19$ lb per sq in. To this stress should be added the shearing stress due to dead load and live load.

In case of an arch with a rigidly connected superstructure (Fig. 5), the arch rib will transmit to the springing lines considerably more stress than the wind load acting upon the rib itself. If the arch is of the barrel type with spandrel walls, a large area is exposed to lateral wind pressure and the arch section, although possessing high resistance to horizontal bending, has comparatively low torsional strength. Shearing stresses for this type of structure will be considerably greater than 19 lb per sq in.

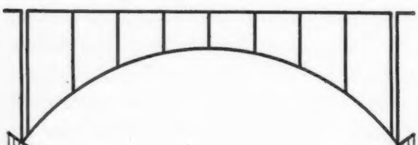

FIG. 5

It is unfortunate that the author chose the symbol, w, to indicate load intensity and dw to represent elastic weight; integration limits in the deflection terms and moment equations would also have been helpful.

[21] Balboa Heights, Canal Zone.

[21a] Received by the Secretary April 17, 1936.

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DISCUSSIONS

SUCCESSIVE ELIMINATION OF UNKNOWNS IN THE SLOPE DEFLECTION METHOD

Discussion

BY L. T. WYLY, M. AM. SOC. C. E.

L. T. WYLY,[27] M. AM. SOC. C. E. (by letter).[27a]—The oft-remarked necessity for ingenuity and resourcefulness in solving a number of simultaneous equations in a statically indeterminate problem is well emphasized by this paper. The author, by expressing in terms of one unknown all other unknowns, arrives at an exact value of each in turn. It is interesting to compare this method of attack with that of successive approximations[28] presented by G. A. Maney, M. Am. Soc. C. E., in his first publication of the slope-deflection relation, and with the moment-distribution method[29] presented by Hardy Cross, M. Am. Soc. C. E. In these latter procedures, the simultaneous solution of the equations is avoided and, by methods of approximations, the unknowns at each joint in succession are determined to any desired degree of refinement, the ultimate values, of course, being exactly the same as those obtained by the author in his exact solution of all equations simultaneously.

In order to compare the relative merits of these different methods of solution of the slope-deflection relation, it may be well to review, briefly, the history of the development of this very useful and general method of analysis. The essential steps are as follows:

(1) The first publication of the slope-deflection relation was in 1893 by Otto Mohr who used the method for the solution of secondary stress problems[30]. The form used by Mohr was limited to this type of problem; that is, to frames without loads between the joints.

(2) In May, 1914, Axel Bendixsen published in Denmark a monograph entitled, "Die Methode der Alpha-Gleichungen zur Berechnung von Rahmen-

NOTE.—The paper by John B. Wilbur, Assoc. M. Am. Soc. C. E., was published in December, 1935, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: March, 1936, by Messrs. C. A. Willson, Paul Andersen, and R. W. Stewart; April, 1936, by Adolphus Mitchell, Jun. Am. Soc. C. E.; and May, 1936, by Messrs. Fang-Yin Tsai, A. Floris, A. W. Fischer, and L. E. Grinter.

[27] With Bureau of Bridges, Div. of Highways, Springfield, Ill.

[27a] Received by the Secretary May 7, 1936.

[28] Eng. Studies, *Bulletin No. 1*, Univ. of Minnesota, March, 1915.

[29] "Analysis of Continuous Structures by Distributing Fixed-End Moments," *Transactions*, Am. Soc. C. E., Vol. 96 (1932), p. 1.

[30] *Zivilingenieur*, December, 1892–January, 1893.

konstructionen", which gives the first general statement of the slope-deflection relation for any type of loading. This publication appears to have been generally overlooked even in Europe until about 1923 when Professor A. Ostenfeld amplified it somewhat[31].

(3) In 1914, Professor Maney evolved and, in March, 1915, published the first general statement of the slope-deflection relation to appear in America[28]. Professor Maney's work was original with him, as at this time he did not know of the work of Mohr and of Bendixsen on this problem. This presentation included any type of loading between joints and expressed the end moments of members in terms of the fixed-end moments, modified by the addition of moments arising from rotations and deflections of the joints. This method was not extended to members with variable moments of inertia at this time. Problems both in wind stresses in building frames and in secondary stresses in bridge trusses were solved; and, for the latter problem, a method of successive approximations was presented, which method did not require the simultaneous solution of the equations.

(4) In 1914, Professor Maney and W. M. Wilson, M. Am. Soc. C. E., developed, and, in June, 1915, published, the solution of wind stresses in a twenty-story office building by the slope-deflection method[32]. This solution was by the exact simultaneous method.

(5) In 1915 to 1918, Professor Maney and J. I. Parcel, M. Am. Soc. C. E., completed their analytical solution and experimental research in the field, and, in 1922, published their study of the secondary stresses on the Kenova Bridge[33]. This solution was made by the use of influence lines for the secondary stresses and involved the solution of forty-three equations for each case considered. The method of successive approximations was again used here in all cases. Eight cases were solved.

(6) In 1918 there was published by Professor Wilson, F. E. Richart, and C. Weiss, Members, Am. Soc. C. E., a bulletin giving general formulas for a large number of types of structures by the slope-deflection method[34].

(7) In 1923 Professor Ostenfeld published an amplification of Bendixsen's work, the presentation covering all cases for which solution is made for deformations rather than stresses[31].

(8) In 1930 Professor Cross published his moment-distribution method which he had been teaching for several years previously[29]. The Cross method also follows the general procedure of assuming fixed-end moments at ends of members and correcting them by a method of approximations that can be carried to any desired degree of refinement.

(9) In 1931, L. T. Evans, Assoc. M. Am. Soc. C. E., published a modification of the slope-deflection formulas, taking into account the variable moment of inertia of members for a number of typical cases[35].

It is thus seen that, in America, a remarkable development has followed Professor Maney's derivation of the general method of slope deflection in 1914.

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[31] "Die Deformationsmethode", by A. Ostenfeld, *Der Bauingenieur*, January 31, 1923.
[32] *Bulletin 80*, Eng. Experimental Station, Univ. of Illinois, June, 1915.
[33] Eng. Studies, *Bulletin No. 4*, Univ. of Minnesota, 1922.
[34] *Bulletin 108*, Eng. Experiment Station, Univ. of Illinois, 1918.
[35] *Journal*, Am. Concrete Inst., October, 1931.

E. J. Mehren, M. Am. Soc. C. E., lists slope deflection and moment distribution as the two analytical instruments which have vitally affected the advance of concrete structures in America[36].

In order to compare the methods of approximate solution with the exact solution by successive elimination of unknowns illustrated by the author, attention is directed to the continuous girder of four spans illustrated in Fig. 1(*a*) of the paper. Assume the loading shown in the illustration, or any other loading. By Maney's method of approximations, the procedure is as follows:

(*a*) Fixed-beam end moments are computed on the assumption that all spans are fixed at the ends. Denote these moments as M_{FAB}, M_{FBC}, M_{FBA}, etc. Let ΣM_{FA} designate the sum of all fixed-beam moments at A.

(*b*) The first approximation of joint rotation values at a given joint are obtained by assuming that all joints at the far ends of the members in question have zero rotation. For purposes of illustration consider Joint B, Fig. 1(*a*). Then,

$$\theta_B = \frac{M_{FBA} + M_{FBC}}{2\,K_{BA} + 2\,K_{BC}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (43)$$

or, in general,

$$\theta_B = \frac{\Sigma M_{FB}}{2\,\Sigma K_B} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (44)$$

Also,

$$\theta_C = \frac{M_{FCB} + M_{FCD}}{2\,K_{CB} + 2\,K_{CD}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (45)$$

or, in general,

$$\theta_C = \frac{\Sigma M_{FC}}{2\,\Sigma K_C} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (46)$$

Substituting these values as found in the joint equations, a new and closer set of values for the joint rotations is obtained, and this can be continued as often as necessary, seldom more than three approximations being necessary.

(*c*) Placing the values of joint rotations as determined by Equations (44) and (46) in the joint equations, solution is made for the resulting end moments. As has been noted, this process consists in adding algebraically the end moments induced by the joint rotations to the fixed-beam end moments assumed at the start.

By the Cross' moment-distribution method of approximations, the procedure is very similar. If the values of joint rotation, obtained by Equations (44) and (46), are substituted in the moment equation for a given member,

$$M_{BC} = M_{FBC} = \left(\frac{K_{BC}}{\Sigma K_B}\right) \Sigma M_{FB} - \frac{1}{2}\left(\frac{K_{BC}}{\Sigma K_C}\right) \Sigma M_{FC} \quad \ldots\ldots (47)$$

[36] "Concrete, Yesterday, Today and Tomorrow", *Journal*, Am. Concrete Inst., March–April, 1935; also, *Proceedings*, Am. Concrete Inst., Vol. 31, p. 345.

It is seen at once that the second term on the right-hand side of the equality sign represents the "distribution factor" and that the third term is the "carry-over" factor. Distribution and carrying-over are continued until all unbalanced moments at the joints become negligible, or disappear, which is a parallel process to arriving at successively closer values of true joint rotations. Indeed, it is noteworthy that the reason the moment-distribution process converges so rapidly, is that a rotation at Joint A, say, will induce only one-half as much end moment at Joint B as at Joint A, etc. This relation is evident at once on inspection of the slope-deflection equation. By Maney's procedure, one solves for closer and closer values of the joint rotations, or deflections, and then obtains final moments from final values of the joint rotations. By Cross' procedure, one distributes moment differences until all unbalanced moments are negligible or disappear. A very interesting development in this regard was made in 1934 by C. E. Morgan, M. Am. Soc. C. E.[37]. Pursuing moment distribution to its logical conclusion mathematically, Mr. Morgan expressed the end moments in terms of a geometric series, and arriving at the limit of the series, obtained the true values of the moments—that is, the same values that would be obtained by Professor Wilbur's exact method of solution. In 1936, Mr. Morgan published formulas for moments for a four-span continuous structure similar to Fig. 1(a). These formulas were derived by Mr. Morgan's series method[37].

For frames such as those illustrated in Fig. 1(b) and Fig. 1(c), the procedure is essentially the same, and the relations between the foregoing methods of attack are similar. Methods of solution by approximations of these frames were published[38] by Professor Maney and J. E. Goldberg, Jun. Am. Soc. C. E., in 1932. In 1931, there was also published[39] a method of solving secondary stresses by moment distribution by L. S. Thompson and R. W. Cutler, Jun. Am. Soc. C. E.

With regard to the method of procedure for a given case, naturally the designer's preference will vary. Where not too many unknowns or joints are involved, it will frequently be desirable to use a procedure similar to that illustrated by the author. Where a large number of unknowns are involved, however, it is the writer's belief that most engineers will prefer one of the approximate methods. It should be noted in this connection that there is a difference in procedure between the slope-deflection method of approximation and the moment-distribution method. In both, any variation in the convergence indicates that an error has been made. In the moment-distribution method, however, it is possible to carry over an initial unbalanced moment of incorrect value, and it will be distributed and no check will appear. In the slope-deflection method, a complete set of static and elastic equations is set up, and when the true values of the unknowns are obtained, by whatever means, these unknowns must satisfy all equations simultaneously, and, hence, must be correct. In many cases the moment distribution method is

[37] "Designing Concrete Girder Bridges for Continuity", *Engineering News-Record*, April 23, 1936, pp. 604–606.

[38] "Simplified Methods for Analysis of Multiple Joint Rigid Frames", *Bulletin No.* 7, Northwestern Univ., October 17, 1932.

[39] *Transactions*, Am. Soc. C. E., Vol. 96 (1932), p. 108.

undoubtedly most expeditious. On the other hand, it is the writer's belief, based upon a certain amount of observation, that there is a tendency for the designer to lose sight of the actual significance of each step of the process with moment distribution, and that this is not so likely to occur with slope deflection. Furthermore, in a number of cases, such as where correction is to be made during construction, it is necessary to know the computed value of the joint rotations or deflections, and in this case slope deflection is plainly necessary.

In conclusion, it may be said that the progressive use of indeterminate structures seems to have been dependent, largely, upon the development of expeditious methods of solving simultaneous equations. This has probably been true—not because of the difficulties of the problem—but because simultaneous solution of a considerable number of equations has required more time than the procedure of most designing offices allows. This paper is of definite value in that it calls attention to means of expediting this analytical process.

AMERICAN SOCIETY OF CIVIL ENGINEERS

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DISCUSSIONS

REINFORCED CONCRETE MEMBERS UNDER DIRECT TENSION AND BENDING

Discussion

BY MESSRS. A. FLORIS, AND DAVID B. HALL

A. FLORIS,[23] ESQ. (by letter).[23a]—In this elaborate paper the author derives formulas for the design of reinforced concrete members subjected to direct tension and bending. He states that the problem has received little attention from authors of engineering texts. This statement is rather broad and not quite correct because the subject is fully covered in several books on reinforced concrete. Suffice it to mention the well-known treatises by Moersch,[24], Manning[25], and others. Perhaps the author has in mind American textbooks in which the design of reinforced concrete structures under the combined action of direct tension and bending has been somewhat neglected. In addition to closed water conduits such a stress condition exists in tanks, silos, bunkers, etc.

The formulas derived by the author are based on the standard method which involves the ratio of modulus of elasticity of steel to that of concrete—the well-known n. It is becoming increasingly evident by tests and actual experience, however, that this method of designing reinforced concrete structures is fundamentally erroneous. The reasons are many and varied.

The strength of a member subjected to bending depends on the ultimate strength of steel and concrete and not on the value of n as the standard method appears to convey. Using this ratio in the calculations, the concrete and steel stresses of the structural member near the breaking point are considerably greater than the ultimate strength of the two materials determined separately. By applying the standard method, authorities permit far smaller stresses for concrete than steel in comparison to their ultimate strength.

NOTE.—The paper by D. B. Gumensky, Assoc. M. Am. Soc. C. E., was published in December, 1935, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: March, 1936, by Messrs. A. W. Fischer, William E. Wilbur, and F. C. Snow; and April, 1936, by Messrs. Ralph E. Byrne, Jr., Carl H. Heilbron, Jr., F. E. Turneaure, H. E. Warrington, William A. Larsen, and B. Kovediaeff.

[23] Dipl.-Ing., Los Angeles, Calif.

[23a] Received by the Secretary, March 30, 1936.

[24] "Der Eisenbetonbau, seine Theorie und Anwendung", von E. Moersch, Stuttgart, 1923, Vol. 1, Pt. 1, pp. 454–466.

[25] "Reinforced Concrete Design", by G. P. Manning, Lond., 1924, pp. 54–67.

This leads to the absurdity that heavily reinforced members will resist external forces less than poorly reinforced members under the same conditions. This conclusion as to the standard method of designing structures is not backed by experience and tests. The introduction, in the analysis, of a constant or variable *n* does not improve matters, since it is not in agreement with the experimental evidence. Furthermore, the factor of safety determined by the standard method is not the same for various conditions.

All these facts indicate clearly the desirability of abandoning the present standard method of designing reinforced concrete structures. Attempts to develop methods without using the ratio, *n*, have been made by Stussi, M. and S. Steuermann, Saliger, Gebauer[26], and Bittner.

DAVID B. HALL,[27] ASSOC. M. AM. SOC. C. E. (by letter).[27a]—Whenever physical conceptions can be substituted for algebraic operations, understanding of any problem usually becomes clearer. Apparently, Case 1 of this paper, in which the resultant tension on a section falls between two layers of reinforcement, could have been covered completely by treating the two layers of steel as simple beam reactions and calculating the area of steel to furnish these reactions at the allowable working stress. The extreme simplicity of this case results from the fact that it is a statically determinate problem, and, therefore, no question of consistent deformations enters into it. Case 2 may also be a little easier to grasp[28] when viewed apart from its mathematical surroundings, and certain aspects can be treated in somewhat greater detail.

When it is possible to proportion a structure (in this case a concrete section) to resist the loads acting on it, instead of determining the stresses in an existing structure, then, as far as the designer is concerned, the structure ceases to be statically indeterminate. (This principle, incidentally, has many and varied applications; for instance, a two-span continuous truss with a jack placed under one support at the time of erection.) This will be the case in the design of concrete sections. It may be possible to vary the dimensions of the concrete, and it will practically always be possible to vary the area of the reinforcement.

When there is only one layer of steel, and moments are taken about the center of the steel, the moment of the concrete must be equal to the moment of all external forces (Equation (20)), and hence the concrete stress can be found either by calculation or from ordinary concrete tables. The unit stress in the steel is proportional to the distance from the neutral axis (Assumption (1) of the paper); and the total stress in the steel is the sum of the compression in the concrete and the external tension, since both these forces tend to push or pull the member apart while the steel tends to pull it together Although, in a sense, the foregoing is only a re-statement of Equations (20) and (21) in words, the fact is that once these principles are

[26] "Das alte *n*-Verfahren und die neuen *n*-freien Berechnungsweisen des Eisenbetonbalkens; Kritische Besprechung und Schlussfolgerungen", von Franz Gebauer, *Beton u. Eisen*, January 20, 1936, p. 29.

[27] Asst. Engr., State Dept. of Public Works, Albany, N. Y.

[27a] Received by the Secretary May 11, 1936.

[28] *Engineering News-Record*, July 26, 1928, p. 127.

understood, the design of sections will actually be found as easy if the analysis stops at this point as if it is carried any further, as exemplified by Fig. 5. The steps, in brief, are as follows: (1) Determine a concrete section to resist the moment about the steel; (2) determine the steel for such a concrete section; and (3) determine also the steel to resist the tension (or compression) on the section, and add to (or subtract from) the steel found by Step (2).

It so happens that it is not always possible to fit the concrete section to the loads. In designing sections at intervals along a beam, many parts will be found to be understressed. On the other hand, in continuous beams with curved bottoms, there is usually an overstressed part near the middle which would receive more moment and would be stressed still more severely if made deeper. Thus, when the depth of the beam is fixed, two distinct cases may arise, depending on whether the moment about the tension steel is less or more than the resisting moment of the concrete in a "balanced section"; that is, one in which the steel and concrete stress are both the maximum allowable. The first case differs little from that in which the concrete as well as the steel can be controlled. It can be solved by means of the author's charts, or directly from the principles underlying Equations (20) and (21).

In connection with the latter method an expedient used by the writer may be found to be helpful: The function least sensitive to the variation of K (defined as $K = \frac{M}{b d^2} = \frac{1}{2} f_c k j$) is j, the fraction of the depth between the steel and the center of compression of the concrete. Since the steel area for moment may be calculated from the relation, $A_s = \frac{M}{f_s j d}$ the writer uses a table of values of $(f_s j)$ for the full range from zero moment to the moment for balanced steel. Since $(f_s j)$ varies only between 16 000 and 14 000 lb per sq in. (when $f_s = 16\,000$ lb per sq in.) in this entire range, a table with twenty entries is as accurate as a table of steel percentage value containing eight or ten times as many entries, and the preliminary determination of K can be very rough.

The second case (in which the strength of the concrete is the limiting factor and must be aided by shifting the neutral axis, adding compression steel, or more often both) is usually beyond the scope of the author's charts, and needs to be considered in some detail if the paper is to be reasonably complete. Unless there are restrictions on the amount of steel to be placed in either face of the beam, the most desirable solution will be the one which makes the total steel area in the two faces a minimum. Contrary to a widespread belief, this condition is usually not fulfilled by assuming maximum steel stress. Two causes can be mentioned which may make this fact easier to digest: (1) In cases of bending and compression, it can be seen that the use of a high steel stress, thereby crowding the neutral axis (and hence the area of concrete in compression) over to one side, will be inefficient whenever there is considerable direct compression and not much moment on a section; and (2) when a beam is not very deep and the compression steel is close to the neutral axis, this steel cannot be very effective. Its effectiveness,

as well as that of the concrete, furthermore, is increased by shifting the neutral axis farther toward the tension side.

In terms of fundamental relations, only one additional operation is necessary in designing for compression steel, after a given position for the neutral axis has been assumed or established. Assumption (1) of the paper together with Hooke's law, establishes the stress in both layers of steel; compressive steel and additional tensile steel are added to provide a couple to care for the difference between the concrete moment and the external moment. With a good comprehensive table, made up for specified values of f_c and n, giving

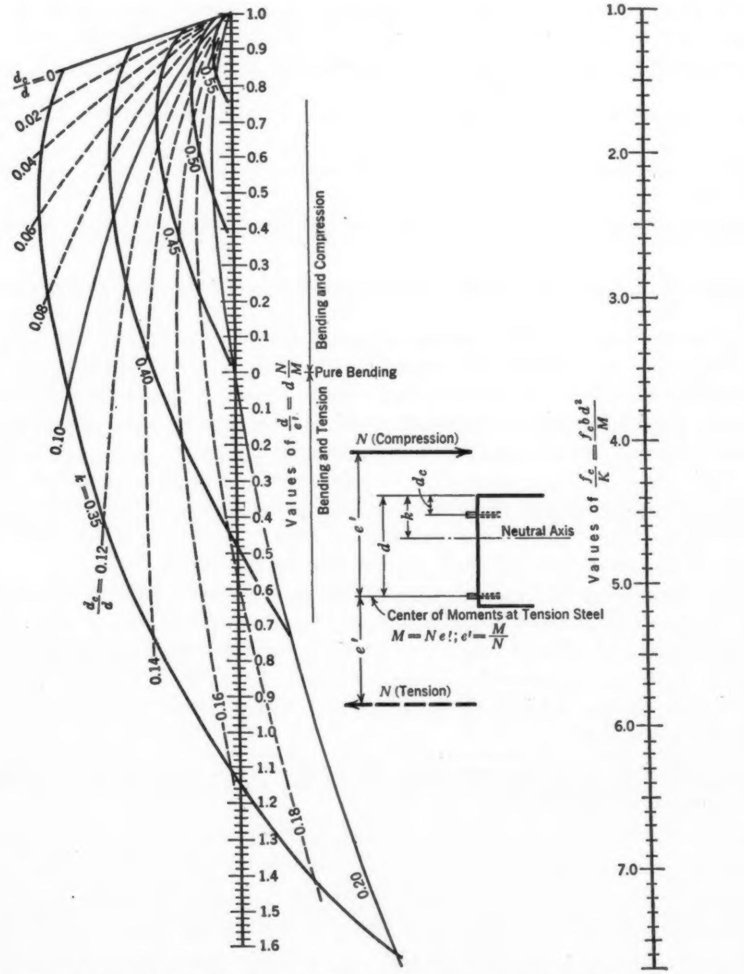

FIG. 11.—POSITION OF NEUTRAL AXIS FOR MINIMUM STEEL AREA. (GIVEN $\frac{f_c}{K}$, $\frac{d}{e'}$, $\frac{d_c}{d}$, THE CURVE GIVES k.

values of k, K, f_s, and p (with k varying by hundredths from the value corresponding to a balanced section up to 0.60 or 0.70), it is not especially difficult to make a few trials and obtain suitable steel areas. Even without such a table the foregoing constants are not very difficult to compute. Instead of by trial, the desired value of k can be found quite expeditiously with the aid of Fig. 11[29], which is self-explanatory.

It will be noted that the three quantities required for its use are all abstract numbers, so that the chart is independent of any particular units of force or distance. For a strict interpretation of Fig. 11 it is necessary to assume that the compression steel displaces no concrete, but the diagram will undoubtedly be found sufficiently accurate by designers who prefer to deduct the concrete (by using $n - 1$ instead of n for the compression steel). The construction of the chart involves considerable algebra, but not much mystery. An equation for the total area of steel can be written in terms of k, $\frac{d_c}{d}$, $\frac{d}{e'}$, and $\frac{f_c}{K}$, and can be differentiated with respect to k. The resulting derivative contains k and $\frac{d_c}{d}$ in a relatively complicated form, but can be arranged to contain $\frac{d}{e'}$ and $\frac{f_c}{K}$ as simple linear quantities. This is what makes the use of a nomogram possible. Examination of Fig. 11 will verify the statements made earlier concerning the causes for shifting the neutral axis. It will be observed: (1) That the compressive area is greater when there is thrust on a section than when there is tension; and (2) it is greater when the compression steel is deeply buried than when it is near the face.

This chart will not always give usable results. It may sometimes call for steel stress higher than the allowable and, being purely algebraic, it can (and sometimes does) lead to the calculation of a negative area of steel, or less than would nominally be used, in one face or the other.

In developing his special type of chart for direct tension and bending, the author has utilized principles which by themselves are as easy to apply as the chart (see Figs. 3 and 5), and which, furthermore, apply equally well to either tension or compression and bending, and may be applied to one important part of the subject of bending and direct stress to which the chart cannot be applied.

[29] "Calcul des armatures principales des pieces en béton armé a section rectangulaire sollictée en flexion composée", by Henri Dumontier, *La Tecnique des Travaux*, Vol. IV, No. 2, February, 1928, p. 135.

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DISCUSSIONS

TALL BUILDING FRAMES STUDIED BY MEANS OF MECHANICAL MODELS

Discussion

By Fang-Yin Tsai, Assoc. M. Am. Soc. C. E.

Fang-Yin Tsai,[12] Assoc. M. Am. Soc. C. E. (by letter).[12a]—In Figs. 3, 5, 7, and 8, of this paper, the curves are plotted with i, or $\frac{L_i}{L_o}$, assumed to be variable as if it were the only factor that governs the value of R_i for the various bents. W. M. Wilson and G. A. Maney, Members, Am. Soc. C. E., have shown[13] that the distribution of moments in columns and girders of a symmetrical 4-column bent depends upon the ratios of the K-values of its members, in addition to the layout and loading. The writer has found that, for bents of the type of the authors' Series A, B, C, and D, the value of R_i depends upon the story number and the following factors:

$$\alpha = \frac{I'_o}{I'_i} \qquad (29)$$

$$\beta = \frac{I'_o}{I_o} \times \frac{L_o}{h} \qquad (30)$$

$$r = \frac{L_i}{L_o} \qquad (31)$$

and,

$$i = \frac{I_i}{I_o} \qquad (32)$$

in which I'_o and I'_i are, respectively, the moment of inertia of the outer and inner column; h is the story height; and the remaining symbols are the same

Note.—The paper by Francis P. Witmer, M. Am. Soc. C. E., and Harry H. Bonner, Esq., was published in January, 1936, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: April, 1936, by Messrs. L. C. Maugh, L. J. Mensch, and Gilbert Morrison.

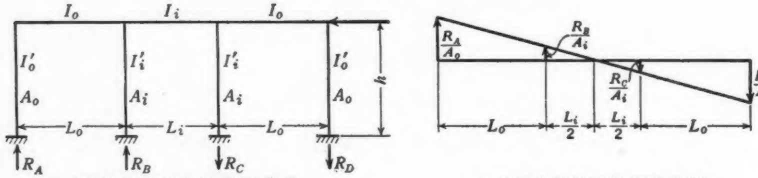
[12] Prof. of Structural Eng., Dept. of Civ. Eng., National Tsing Hua Univ., Peiping, China.

[12a] Received by the Secretary April 6, 1936.

[13] "Wind Stresses in the Steel Frames of Office Buildings", *Bulletin No. 80*, Eng. Experiment Station, Univ. of Illinois, 1915, p. 26.

as those used in the paper. The values of α, β, r, and i. together with the story number, may be considered as the elastic properties of such bents, which govern the value of R_i due to lateral loads.

An attempt has been made to verify the authors' results analytically and to deduce a general expression for R_i in terms of α, β, r, and i. Since such work for a 19-story, 4-column bent involves a prohibitive amount of time and difficulty, the writer selected an expression for such bent of one-story only (Fig. 17(a)). (He is indebted to several students in his class in Advanced



(a) 1-STORY 4-COLUMN SYMMETRICAL BENT (b) CANTILEVER RELATION OF REACTIONS

FIG. 17

Structural Theory for developing the formula by the slope-deflection method.) The result is as follows:

$$R_i = \frac{2\,i\,(2\,\beta - \alpha + 2)}{r^2(4\,\beta + \alpha + 1) + 3\,\alpha\,r\,i} - 1 \quad \text{.............(33)}$$

The elastic properties and the authors' values of R_i for the 19-story bents of Series A, B, C, and D, together with the computed values of R_i for one-story bents of the same elastic properties, are given in Table 2, in which it is

TABLE 2.—COMPARISON OF THE VALUES OF R_i OBTAINED BY THE AUTHORS FOR 19-STORY BENTS WITH THOSE COMPUTED FOR ONE-STORY BENTS OF THE SAME ELASTIC PROPERTIES

Case	Elastic Properties of Bents				Values of R_i, in Percentages		Case	Elastic Properties of Bents				Values of R_i, in Percentages	
	α	β	r	i	The authors' (for 19-story)	Computed (for one-story)		α	β	r	i	The authors' (for 19-story)	Computed (for one-story)
A-1....	1	1.6	1	1	− 15	− 26.4	B-1....	1	1.2	2	1	− 76	− 79.5
A-2....	1	1.6	1	2	+ 33	+ 16.7	B-2....	1	1.2	2	2	− 46	− 65.3
A-3....	1	1.6	1	3	+ 89	+ 44.9	B-3....	1	1.2	2	3	− 33	− 54.9
A-9....	1	1.6	1	9	+169	+113.5	B-9....	1	1.2	2	9	+ 13	− 24.6
A-17...	1	1.6	1	17	+230	+140.0	B-17...	1	1.2	2	17	+ 46	− 10.5
A-1a...	1	1.6	1	1	− 19	− 26.4	B-1a...	1	0.15	2	1	− 56	− 84.2
A-1b...	8	12.8	1	1	− 33	− 68.4	C-1....	1	1.8	0.67	1	+ 76	+ 51.0
A-1c...	0.125	1.6	1	1	− 3	+ 28.5	C-2....	1	0.9	0.67	0.5	± 0	− 19.8
D-1....	1	0.8	1	1	− 18	− 36.6	XC-1..	1	1.72	0.78	1	+ 29	+ 14.6
D-2....	1	0.8	1	2	+ 21	− 7.2	....	....	...	..	..	.....	.....
D-3....	1	0.8	1	3	+ 46	+ 9.9	....	....	...	..	..	.....	.....
D-9....	1	0.8	1	9	+ 56	+ 45.4	....	....	...	..	..	.....	.....

seen that, with the exception of Case A-1c, all the computed values of R_i are consistently smaller (algebraically) than the corresponding values obtained by the authors. Both sets of the values of R_i are plotted against the values of

i for Series A, B, and D in Fig. 18, which is similar to the authors' Fig. 3. It shows, not only the consistency in the trends of the curves and the correctness of the authors' results in general, but also that, with α, β, and r kept constant, the values of R_i increase algebraically as the story number increases

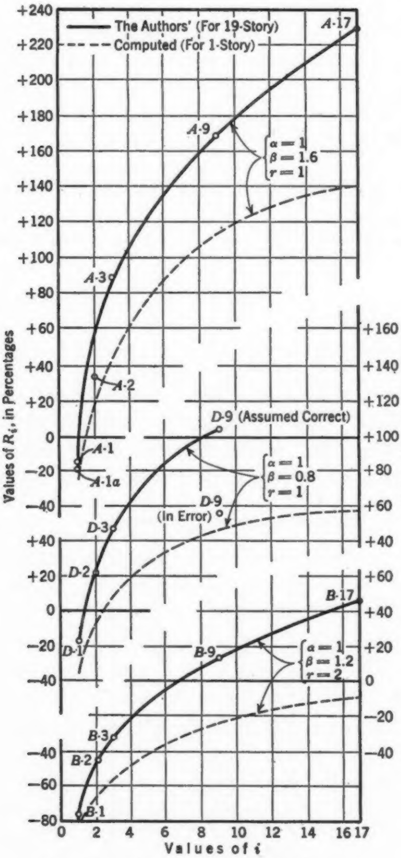


FIG. 18.—RELATION BETWEEN R_i AND i

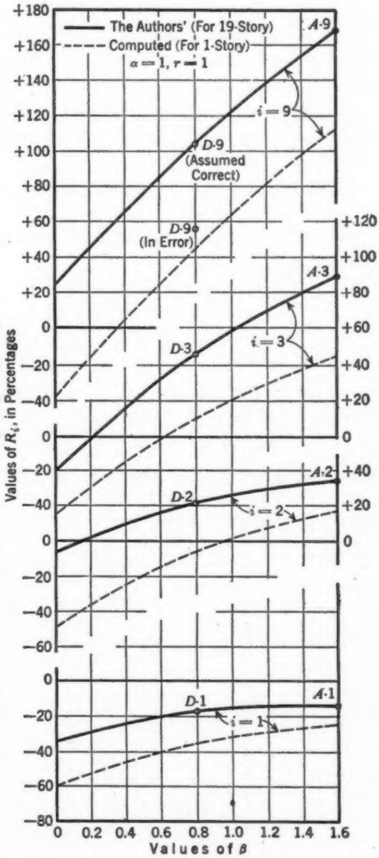


FIG. 19.—RELATION BETWEEN R_i AND β

for any value of i, and that the larger the value of i, the greater will be the increase in the value of R_i.

Fig. 19 shows the relation between R_i and β, with α, r, and i kept constant, comparing the authors' values for the 19-story bent with those computed for the one-story bent. It is seen that, for this case, the values of R_i increase algebraically as the story number increases for any value of β, and that the larger the value of i, the greater the increase in the value of R_i.

The authors' Fig. 8 shows the relation between i and r for the cantilever relation of R_i, neglecting again the other elastic properties of the bents.

From Fig. 17(*b*), the cantilever relation of reactions is governed by;

$$\frac{\dfrac{R_B}{A_i}}{\dfrac{R_A}{A_o}} = \frac{\dfrac{L_i}{2}}{L_o + \dfrac{L_i}{2}} \qquad (34)$$

or,

$$R_i = \frac{R_B}{R_A} = \frac{q\,r}{2 + r} \qquad (35)$$

in which $q = \dfrac{A_i}{A_o}$ = the ratio of the area of the inner column to that of the outer column. Equating Equation (35) to Equation (33) and simplifying,

$$i = \frac{q\,r^2\,(1 + r)\,(4\,\beta + \alpha + 1)}{(2\,\beta - \alpha + 2)\,(2 + r) - 3\,q\,\alpha\,r\,(1 + r)} \qquad (36)$$

Letting $\alpha = q = 1$, which is the value for all the bents (except Cases *A*-1*b* and *A*-1*c*) of the authors' Series *A*, *B*, *C*, and *D*, Equation (36) is plotted with β = 0.8; 1.2; 1.6; and 2.5 in Fig. 20, in which it is seen that the curve with β = 2.5 coincides practically with the authors' Fig. 8. Fig. 20 also

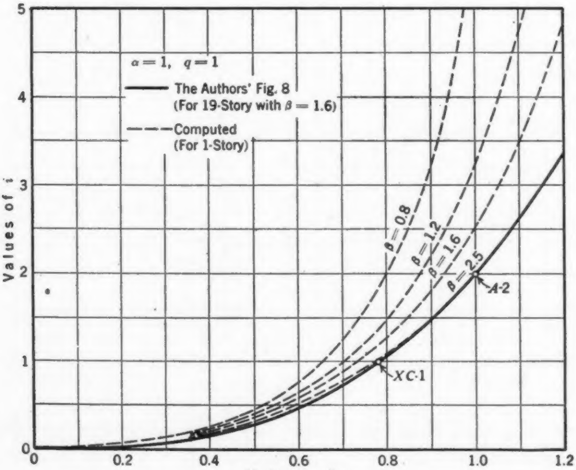


FIG. 20

shows that, for a cantilever-theory value of R_i in one-story bents, with $\alpha = q = 1$, i decreases as β increases for any value of r, and that with β kept constant, i increases rapidly as r increases. Assuming the authors' Fig. 8 to be valid for 19-story bents with β = 1.6, the increase in the story number will have the same effect on i as the increase in β, and the girder

length of Case XC–1 should be adjusted to make $\beta = 1.6$. Thus, by Equation (30),

$$\beta = 1.6 = \frac{I'_o}{I_o} \times \frac{L_o}{h} = \frac{I_o}{2.5} \quad \text{................(37)}$$

in which $I'_o = I_o$. Hence, $L_o = 1.6 \times 2.5 = 4$ in.; and $L_i = i\ L_o = 0.78 \times 4 = 3.12$ in. If the foregoing values were used for L_o and L_i in Case XC–1 instead of $4\frac{5}{16}$ and $3\frac{3}{8}$ in., the value of R_i would probably be much closer to the cantilever-theory ratio than that obtained by the authors.

For a portal-theory value of R_i in one-story bents, equating Equation (33) to zero and simplifying:

$$i = \frac{r^2\ (4\ \beta + \alpha + 1)}{4\ (\beta + 1) - \alpha\ (2 + 3\ r)} \quad \text{................(38)}$$

In order to avoid any negative value in i, which will be meaningless, α must always be less than $\frac{4\ (\beta + 1)}{2 + 3\ r}$.

The purpose of the foregoing discussion is mainly to verify the authors' results by comparing them with those computed for one-story bents of the same elastic properties and also to indicate the trend in the effect of the various factors, particularly the story number, upon the value of R_i.

The authors state that "the purpose of the study was to learn trends rather than true quantitative values of reactions for varying proportions of bents." Inasmuch as the value of R_i depends upon the numerous factors indicated herein, and the layout and the proportion of bents in any practical case will differ widely from those studied in the paper, it is doubtful whether the trends of any actual bent will conform to those presented by the authors. The writer thinks that the real value of the paper lies in introducing a simple and practical method of study, which, when applied carefully to an actual bent, may eventually lead to its proper proportion and design.

The writer wishes to call the authors' attention to the following minor points. The letters, I and G, are both used for denoting the moment of inertia of girders, and this is evidently due to an oversight. The authors use the title, "Reaction Diagram" for their Figs. 2, 4, etc., but the writer thinks that the diagrams will be better understood if the title is changed to read, "Reaction Influence Diagram." In the authors' Fig. 8 there are indicated Cases B–18 and C–0.67, both of which are apparently not described in the paper.

From the trends of the curves in Figs. 18 and 19, the writer concludes that the authors' value of R_i for Case D–9 is inaccurate, possibly due to the great inflexibility of the center girders of the bent. This is also indicated by the great asymmetry of the authors' Fig. 4(d). Hence, the curve is plotted according to an assumed correct value of R_i for Case D–9.

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DISCUSSIONS

MODERN CONCEPTIONS OF THE MECHANICS OF FLUID TURBULENCE

Discussion

BY JOE W. JOHNSON, JUN. AM. SOC. C. E.

JOE W. JOHNSON,[33] JUN. AM SOC. C. E. (by letter).[33a]—An extremely timely summary of the modern conceptions and theories of the mechanics of fluid turbulence is offered in this paper. The presentation is clear and orderly and contains a consistent and logical system of notation. The application of these principles affords new methods of approach to many complex hydraulic problems.

One such application, for example, is the problem of silt transportation advanced by W. Schmidt[34] and outlined by Morrough P. O'Brien[35], Assoc. M. Am. Soc. C. E., in 1932, in a review of the theory of turbulent flow and its relation to silt transportation in open channels. In later papers, Messrs. J. B. Leighly[36] and J. E. Christiansen[37] discussed Professor O'Brien's paper and reported encouraging results in comparing measured silt distribution and theory. The silt referred to is the solid matter transported in suspension, but not that which moves as bed-load. It is recognized, of course, that there is no dividing line between the classifications. Therefore, in view of the fact that the difference between suspended load and bed-load is purely arbitrary, it is fairly reasonable to believe that the theories discussed by the author and by Professor O'Brien may ultimately provide an adequate and accurate relation between bed-load transportation and certain hydraulic elements.

NOTE.—The paper by Hunter Rouse, Assoc. M. Am. Soc. C. E., was published in January, 1936, *Proceedings*. Discussion on the paper has appeared in *Proceedings*, as follows: April, 1936, by Chesley J. Posey, Jun. Am. Soc. C. E.; and May, 1936, by Messrs. S. Franz Yasines, Benjamin Miller, and Ralph W. Powell.

[33] Jun. Hydr. Engr., Sedimentation and Hydraulic Studies, Soil Conservation Service, U. S. Dept. of Agriculture, Washington, D. C.

[33a] Received by the Secretary April 10, 1936.

[34] "Die Massenaustauch in freien Luft und verwanule Erscheinungen", von W. Schmidt, H. Grand, Hamburg, 1925.

[35] "Review of the Theory of Turbulent Flow and Its Relation to Sediment Transportation", by Morrough P. O'Brien, *Transactions*, Am. Geophysical Union, Hydrology Section, 1933.

[36] "Turbulence and Transportation of Rock Débris by Streams", by J. B. Leighly, *The Geographical Review*, Vol. XXIV, No. 3, July, 1934.

[37] "Distribution of Silt in Open Channels", by J. E. Christiansen, *Transactions*, Am. Geophysical Union, Hydrology Section, 1935, p. 478.

The Prandtl-von Kármán theory of velocity distribution in turbulent flow can also be applied to open channels of great widths where the lines of equal velocity are parallel to the channel bed and viscous resistances are assumed to be negligible. Equation (52) as developed for pipes, is a general equation for turbulent flow in either tubes or open channels. For open channels, the expression for the distribution of velocity in the central region of the flow is:

$$\frac{v_{\max} - v}{\sqrt{\frac{\tau_o}{\rho}}} = -\frac{1}{\kappa}\left[\log_e\left(1 - \sqrt{1 - \frac{y}{d_o}}\right) + \sqrt{1 - \frac{y}{d_o}}\right] \ldots\ldots (83)$$

or,

$$\frac{v_{\max} - v}{\sqrt{\frac{\tau_o}{\rho}}} = \phi_1\left(\frac{y}{d_o}\right) \ldots\ldots (84)$$

in which τ_o is the intensity of shear at the channel bottom; d_o, the depth of flow, or the distance from the bottom to the point of maximum velocity when the maximum velocity is below the water surface; and y, the distance of the velocity measurement, v, from the channel bottom (see Fig. 23).

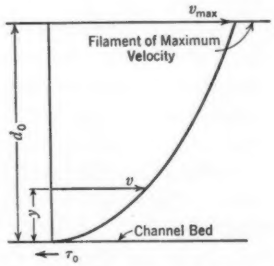


FIG. 23.—VERTICAL VELOCITY CURVE IN OPEN CHANNEL.

Equation (84), derived without regard to wall conditions, gives a theoretical velocity distribution only in the central regions of flow. Another equation must now be derived for the velocity distribution near the wall. Substitution of Prandtl's mixing length as given by the relation,

$$l = \kappa\, y \ldots\ldots (85)$$

into Equation (44) gives the expression,

$$\frac{dv}{dy} = \frac{1}{\kappa\, y}\sqrt{\frac{\tau_o}{\rho}}\sqrt{\frac{d_o - y}{d_o}} \ldots\ldots (86)$$

Integration of Equation (86) with the condition that $v = 0$ when $y = 0$, gives,

$$\frac{v}{\sqrt{\frac{\tau_o}{\rho}}} = \frac{2}{\kappa}\left[4 + \sqrt{1 - \frac{y}{d_o}} - \tan h^{-1}\sqrt{1 - \frac{y}{d_o}}\right] \ldots\ldots (87)$$

or,

$$\frac{v}{\sqrt{\frac{\tau_o}{\rho}}} = \phi_2\left(\frac{y}{d_o}\right) \ldots\ldots (88)$$

Of primary interest to the hydraulic engineer, of course, is the agreement between the foregoing theoretical formulas and experimental data. For illustration in this respect, vertical velocity curves will be used, that were measured by the writer in connection with research work at the U. S. Waterways Experiment Station, at Vicksburg, Miss. These experiments consisted of complete velocity traverses at a section near the center of a tilting rectangular flume, 45 ft long, 2.313 ft wide, and 1.3 ft deep, with smooth cement lining. The flume was given slopes of 0.0002, 0.0005, 0.0010, 0.0015, 0.0020, and 0.0040. At all times the water-surface slope was maintained parallel to the channel bottom. Velocities were measured by an accurately calibrated Pitot tube, which was mounted on the lower end of an ordinary laboratory gage which, in turn, was mounted on a rack that traveled upon a horizontal bar. The horizontal bar support was graduated along the top side to 0.01-ft intervals, and the laboratory gage read to 0.001 ft. With this precise horizontal and vertical control, it was possible to place the Pitot tube at identical locations for all tests and check measurements. The number and location of velocity observations in a typical cross-section are illustrated by Fig. 24. Average vertical velocity curves in the center of the flume for depths of 0.10 ft and 0.20 ft and for various slopes are shown in Fig. 25.

As stated by the author in referring to straight tubes, it was an important discovery that the relative motion of the particles at moderately large Reynolds numbers depends on the fall in pressure and not at all on the character of the wall, so that, therefore, with constant fall in pressure, the velocity distribution curves in tubes with different values of relative roughness can be brought into conformity by shifting them along the velocity axis. The single curve resulting from superimposing the central parts of the curves in Fig. 15(*a*) of the paper is evidence of this fact; but it is not true, of course, for the layer in immediate contact with the wall, where the velocity increase is greater on a smooth surface than on a rough one. Comparable to this for open channels would be the vertical distributions in channels of infinite width in which surface roughness, depth of flow, and the slope of the energy gradient have been adjusted so that the bottom shearing force given by $\gamma\, d_0\, S$ is a constant. In channels of finite width, however, the question of

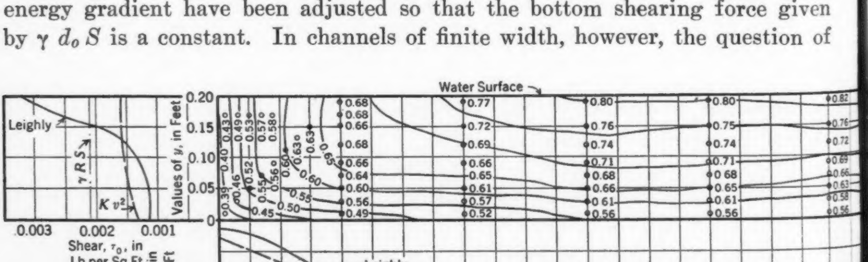
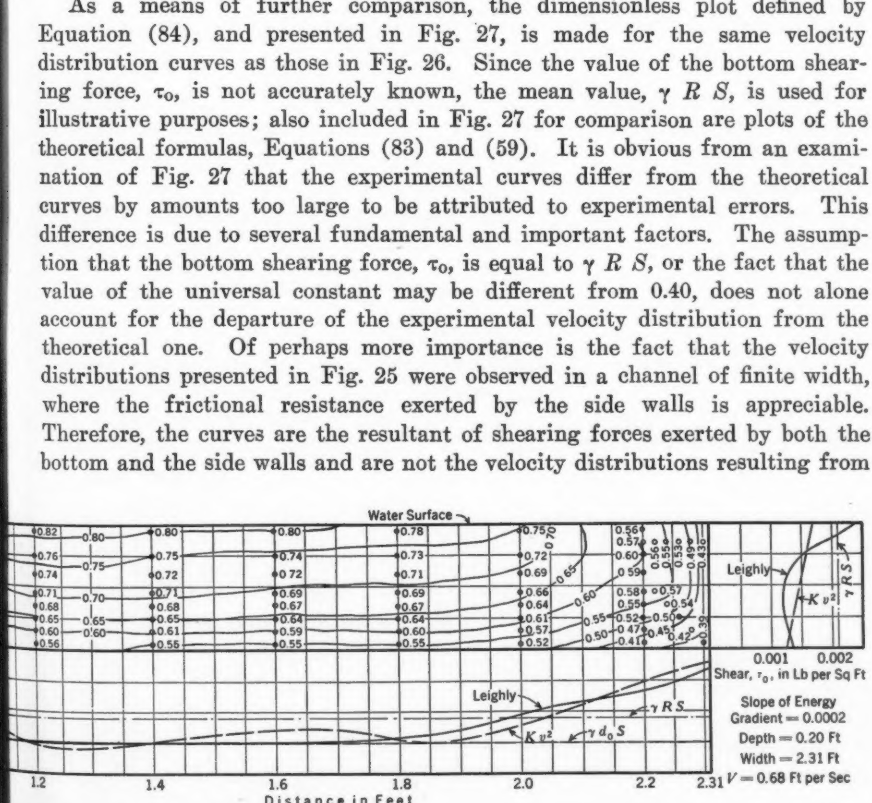


FIG. 24.—DISTRIBUTION OF VELOCITY A

the effect of shearing force along the side walls, as well as the bottom, becomes important. In this case the bottom shear is no longer equal to $\gamma\ d_0\,S$. Where flow resistance at the side walls is appreciable, the average shearing force on the side walls and bottom is given by $\gamma\ R\ S$, in which R is the hydraulic radius and S = slope of the energy gradient. As illustrated by Fig. 24, the shearing force on the bottom exceeds the average value in most cases and approaches, but should not equal, $\gamma\ d_0\,S$. A dimensionless plot similar to Fig. 15(a) is not attempted in this discussion, but a plot of velocity distribution is presented (Fig. 26) for those cases in which the bottom shearing force, as given by equal values of $\gamma\ d_0\,S$, is approximately constant. These velocity distribution curves differ slightly due to the fact, no doubt, that the bottom shearing forces may be (and are, perhaps, not) constant for any two pair of curves (analogous to this in pipe flow would be the condition in which fall in pressure is not constant); nevertheless, by superimposing those curves in which the values of the bottom shearing forces are approximately equal, it is evident that similar flow conditions will occur in the central region of the flow when the value of the bottom shearing force is a constant.

As a means of further comparison, the dimensionless plot defined by Equation (84), and presented in Fig. 27, is made for the same velocity distribution curves as those in Fig. 26. Since the value of the bottom shearing force, τ_0, is not accurately known, the mean value, $\gamma\ R\ S$, is used for illustrative purposes; also included in Fig. 27 for comparison are plots of the theoretical formulas, Equations (83) and (59). It is obvious from an examination of Fig. 27 that the experimental curves differ from the theoretical curves by amounts too large to be attributed to experimental errors. This difference is due to several fundamental and important factors. The assumption that the bottom shearing force, τ_0, is equal to $\gamma\ R\ S$, or the fact that the value of the universal constant may be different from 0.40, does not alone account for the departure of the experimental velocity distribution from the theoretical one. Of perhaps more importance is the fact that the velocity distributions presented in Fig. 25 were observed in a channel of finite width, where the frictional resistance exerted by the side walls is appreciable. Therefore, the curves are the resultant of shearing forces exerted by both the bottom and the side walls and are not the velocity distributions resulting from



SHEARING FORCES IN RECTANGULAR FLUME.

shearing forces exerted only by the bottom. It is to be anticipated that frictional forces exerted by side walls tend to present experimental curves that are more erect than theoretical curves.

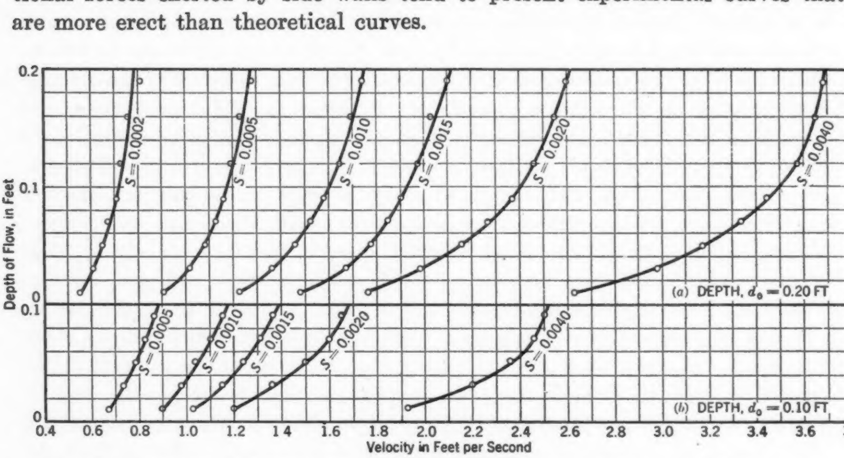


FIG. 25.—VERTICAL VELOCITY CURVES ALONG THE CENTER LINE OF A RECTANGULAR FLUME 2.31 FEET WIDE.

Although not considered in the foregoing discussion, it is of primary importance to recognize that the energy of water flowing in open channels is not dissipated by frictional resistances only. In the general case energy is used in several different ways, the most important of which are: (*a*) In overcoming frictional resistances of drag along the channel walls; (*b*) in overcoming resistances due to purely viscous shearing forces; (*c*) in transporting

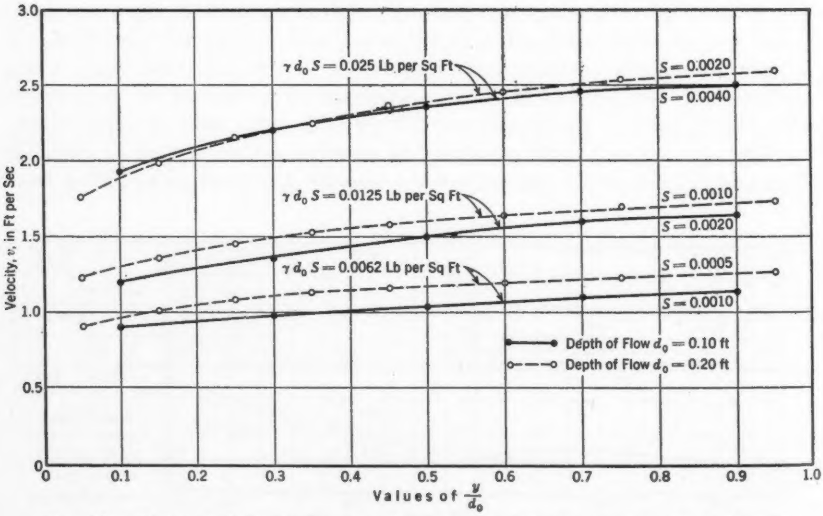


FIG. 26.—VELOCITY DISTRIBUTION IN OPEN CHANNELS WHERE BOTTOM SHEARING FORCES ARE APPROXIMATELY CONSTANT.

solids; (*d*) in the alteration of the channel cross-section; (*e*) in creating and maintaining waves on the surface of the stream; and (*f*) in creating and maintaining sand ripples. Although Factor (*a*) is perhaps the most important

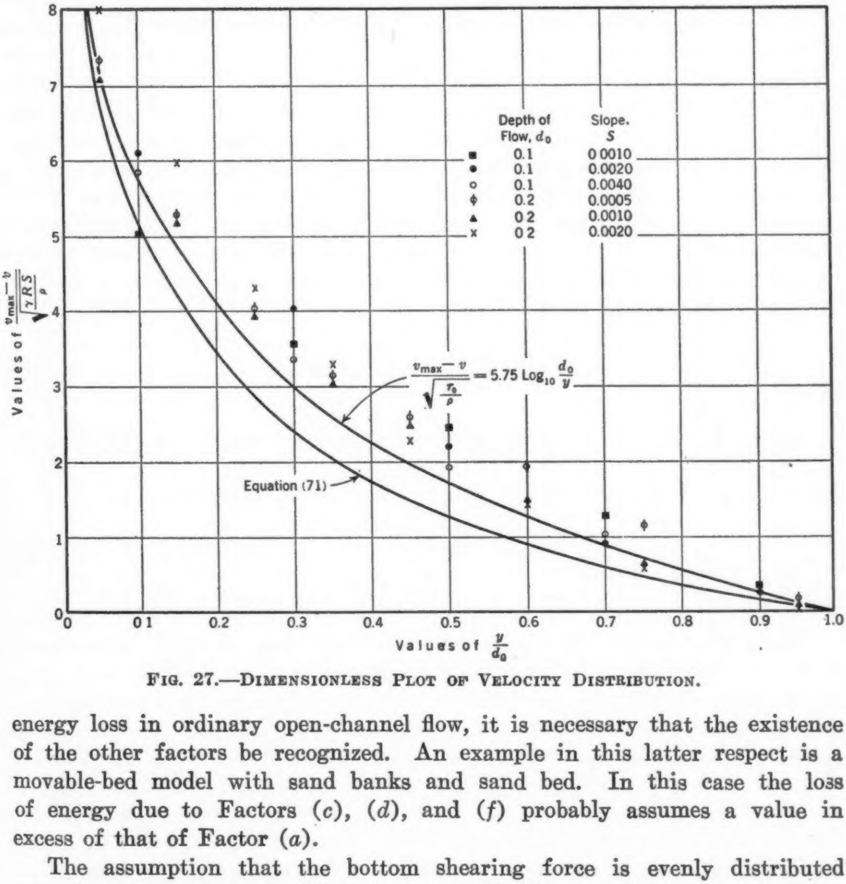


FIG. 27.—DIMENSIONLESS PLOT OF VELOCITY DISTRIBUTION.

energy loss in ordinary open-channel flow, it is necessary that the existence of the other factors be recognized. An example in this latter respect is a movable-bed model with sand banks and sand bed. In this case the loss of energy due to Factors (*c*), (*d*), and (*f*) probably assumes a value in excess of that of Factor (*a*).

The assumption that the bottom shearing force is evenly distributed over the wetted perimeter (that is, $\tau_0 = \gamma\ R\ S$) is merely an approximation as is shown by the irregularities in the isovels. That the trace of the isovels in a stream cross-section and the distribution of shearing force over the wetted perimeter are inextricably related was recognized by Leighly[38]. In 1932, he described a European method of approximating the distribution of shearing force, which consists in dividing the wetted perimeter into equal lengths and drawing from the dividing points between these areas, lines, perpendicular to the equi-velocity lines. The areas bounded by the perimeter, the line of maximum velocity, and the orthogonal lines are proportional to the shear at the bottom. The result of such an analysis for the cross-section

[38] "Toward a Theory of the Morphologic Significance of Turbulence in the Flow of Water in Streams", by J. B. Leighly, Univ. of California, *Publications in Geography*, Vol. 6, No. 1, 1932, Univ. of California Press, Berkeley, Calif.

shown in Fig. 24 is plotted below the section. For comparison, a plot of the average distribution of shear, as given by $\gamma\ R\ S$, and the bottom shear, as given by $\gamma\ d_0\ S$, are also given. Another approach to approximating the distribution of shear along the wetted perimeter is from the consideration that the shear is proportional to the square of the velocity measured at a distance, y, from the wall; that is, τ_0 varies as $K\ v^2$, in which K is a function of the distance of the velocity observation, v, from the wall. From this consideration, a picture of the distribution of shearing force in an open channel can be built up by first observing the magnitude of the velocity at a constant distance, y, from the wall at various locations over the wetted perimeter of a section. Values of v^2 are then plotted along lines drawn perpendicular to the channel wall at the respective points of observation, the plotted points are connected by a smooth curve, and the area between the curve and the channel wall is obtained by planimeter (see Fig. 24). A mean value of K for the section is obtained by the relation,

$$\gamma\ A\ S = K \int_0^p v^2\, dp \qquad (89)$$

in which $\int_0^p v^2\, dp$ is the area under the v^2 mass curve, p is the wetted perimeter, and A is the area of the water cross-section; therefore,

$$K = \frac{\gamma A S}{\int_0^p v^2\, dp} \qquad (90)$$

The multiplication of the values of the ordinates of the v^2 mass curve by K gives the value of the shearing force at the point in question. It is to be noted, however, that the value of K is only an average for a specific case since it varies with wall roughness, depth, and slope, and may also vary from point to point along the wetted perimeter. Calculation of the distribution of shear force for the section in Fig. 24 by the foregoing method yields a distribution that compares quite well with the distribution obtained by applying Leighly's method. Of particular interest is the variation in shearing force distribution resulting when various depths and slopes were introduced into the flume, calculation of shearing force being made by the v^2 mass-curve method. The total shearing force exerted by the side walls is equal to the areas under those parts of the shearing-force distribution diagram which have the side walls as bases. The percentage of the total shearing force exerted by the two side walls was found to vary with depth and to be a constant for a particular depth, regardless of slope. Fig. 28 shows a plot of the percentage of shearing force exerted by the side walls for various depths of flow in the flume.

The information presented in Fig. 28 gives a picture of the magnitude of the error introduced when it is assumed that the total energy of a flowing stream is dissipated in frictional resistances along only the channel bottom. This erroneous assumption is particularly serious in flume studies of the

transportation of bed-load material. The usual procedure in such work has been to attempt to correlate the rate of bed-load movement (in pounds per hour per foot width of flume) with the tractive force (in pounds per square foot) as calculated by the term, $\gamma\ d_0\ S$.

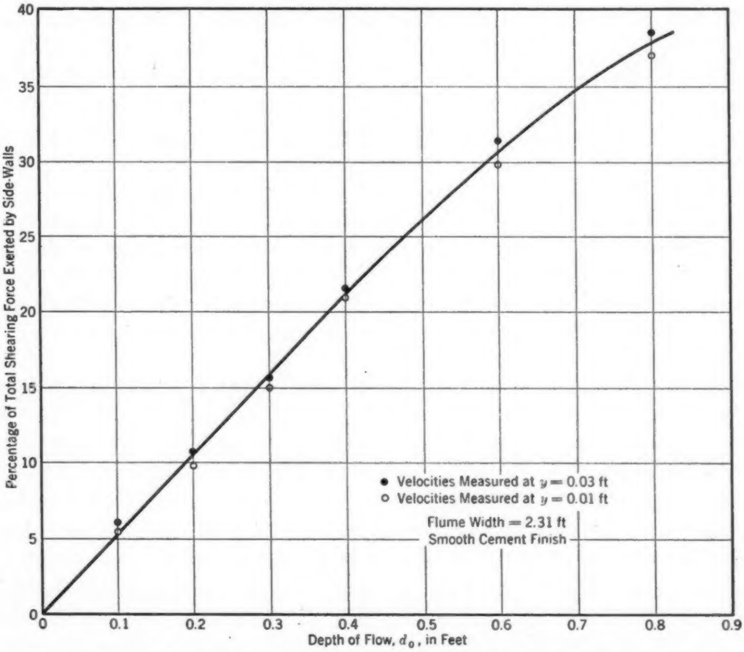


FIG. 28.—RELATION OF DEPTH OF FLOW AND SHEARING FORCE EXERTED BY SIDE-WALLS.

Examination of Fig. 28 shows clearly that the difference between the true value of the bottom shearing force and the value of $\gamma\ d_0\ S$ varies with the depth of flow. Of particular importance is the fact that for a certain depth, d_0, this difference becomes greater than is shown in Fig. 28 if the bottom roughness is made rougher than the side walls; that is, for example, in the case of the sand bed in the ordinary, smooth-wall, experimental flume the bed roughness increases with increased height of sand ripple, the result being an increase in the percentage of the total shearing force exerted by the side walls (depth and slope, of course, are held constant as sand ripples change form). In his experiments on the transportation of bed-load, Hans Kramer, M. Am. Soc. C. E.[39], attempted to correct the condition of unequal bottom and wall roughness by roughening the side walls artificially with sand sprinkled on a coat of wet oil paint to which the grains adhered after the paint dried. Although this procedure is fundamentally sound, it is doubtful whether this type of roughness is effective in reducing the percentage of the shearing force that is exerted on the side walls to the value where the bottom shear can be

[39] "Sand Mixture and Sand Movement in Fluvial Models", by Hans Kramer, *Transactions*, Am. Soc. C. E., Vol. 100 (1935), p. 798.

considered equal to $\gamma\ d_0 S$ as in a channel of infinite width. In this case the use of the average shearing force, as given by the term, $\gamma\ R\ S$, perhaps, gives a better approximation of the value of the bottom shear. Another method of reducing the effect of frictional resistances at the side walls is an arrangement used by C. H. MacDougall[40] for diverting the sand and the water which flowed adjacent to the walls of the channel. This was done by two longitudinal plates fixed 4 in. from each side of the flume at the outlet tank. By this means it was possible to separate the sand and water which moved down the central part of the flume and exclude from the measurement that sand and water which had been retarded by the wall friction. The results of the experiments were assumed to be those which might have been found in a channel of infinite width. However, from the foregoing discussion, it is obvious that such an assumption for an experimental flume is perhaps not true. Perhaps the neglect of the fact that a large percentage of the energy of a stream may be expended at the side walls of a channel, and, therefore, may not be useful for moving bed-loads, accounts for the erratic variations and inconsistent relations obtained in many experiments on the transportation of bed-load.

From the foregoing considerations, it seems that experiments to determine a relation between channel-wall shearing force and velocity distribution for channels of both finite and infinite width would afford a rich and desirable field for research. H. Engels[41] performed some experiments in this respect by measuring the tangential drag on a sand-coated plate 50 by 10 cm in size, recessed in the channel bottom. Simultaneous with the drag observation, the vertical velocity gradient at the center of the plate was also measured. This work has been criticized because the channel was not roughened artificially, as was the plate, and the assumption that the shearing force on the plate is equal to the average shear is probably not strictly true; furthermore, the velocity distribution is probably changing along the plate, thus causing a non-uniform distribution of shear. Simultaneous observation of velocity gradient by accurate Pitot tubes and of distribution of shearing force with an instrument perhaps of the type recently used by Townsend[42] would prove of great value to a better understanding of the mechanics of flow in open channels.

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[40] "Bed Sediment Transportation in Open Channels", by C. H. MacDougall, *Transactions*, Am. Geophysical Union, 1933.

[41] "Versuche über den Reibungswiderstand zwischen stromenden Wasser und Bettsohle", von H. Engels, *Zeitschrift für Bauwesen*, p. 473, 1912.

[42] For description, see "Aerodynamic Theory", Vol. 111, Pt. G. W. F. Durand, Editor. (Published in English by Julius Springer, Berlin, 1935.)

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DISCUSSIONS

BEHAVIOR OF STATIONARY WIRE ROPES IN TENSION AND BENDING

Discussion

BY INGVALD E. MADSEN, JUN. AM. SOC. C. E.

INGVALD E. MADSEN,[12] JUN. AM. SOC. C. E. (by letter).[12a]—The assumption is made in this paper that the loss of strength of wire ropes over sheaves is practically all due to bending stresses, and thus may be evaluated by some bending stress formula. The writer believes that this assumption is not entirely justified, because all bending stress formulas are based on elastic conditions, and, consequently, cannot apply with any accuracy above the elastic limit, and certainly not at the breaking loads. Most of the usual stress theories, apparently, give absurd stresses for the breaking loads, and, consequently, have been disparaged as not picturing true conditions. However, these theories actually may not be so far wrong within the elastic limit. This would seem to be borne out by the breaking of ropes in service under comparatively low loads, because the bending of the rope over sheaves causes stresses far above the fatigue limit, and it takes a relatively small number of repetitions of load to cause some of the wires to break.

The same conditions are present in the usual structural research problems. In order to determine the stresses in a structure, experimentally, the strains are measured and multiplied by the modulus of elasticity of the material (usually 30 000 000 lb per sq in. for steel). However, if the material has exceeded its yield point, large strains are present, and if the method is then applied, calculated stresses result which are far greater than the breaking stress for the material. Analogously, the same thing occurs when the stress theories for bending in wire rope, are extrapolated to the breaking loads.

Actually, when a wire is bent over a sheave, large bending stresses occur at first; but as soon as the yield point of the wire is reached the wire yields, resulting in a re-adjustment of stress. Since practically all the wires used in cables have no definite yield point, this is a gradual process.

NOTE.—The paper by Douglas M. Stewart, Jun. Am. Soc. C. E., was published in February, 1936, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: May, 1936, by Messers. C. D. Meals, and G. P. Boomsliter.

[12] Medford, Mass.

[12a] Received by the Secretary April 29, 1935.

There are several other causes that will weaken a rope when tested over a sheave. One of the most important is the nicking effect which occurs between the outside wires of the strands. As the strands are wrapped around the center core, the outside wires of the strands bear on each other, and since these individual wires cross each other at a fairly sharp angle on the inside of the strand, they nick each other as the rope compresses under load. These nicks reduce the cross-sectional area of the individual wires, and, consequently, weaken the wire. When a rope is bent over a sheave, the nicking effect is more pronounced, since the bottom of the rope bears on the sheave, and the normal force between the wires will be augmented by the radial force of the rope on the sheave and its reaction. This radial force[13] is equal to the tension in the rope divided by the radius, or, in other words, is inversely proportional to the diameter of the rope. No nicks of any magnitude occurred between the wires of the same strand, since these wires cross each other at a small angle, and have a long bearing surface on each other.

Thus, the wires which will break first are not the inside wires as has been usually assumed, but the outside wires of the strands which break at the nicks formed by the crossing wires. These breaks are not visible to the eye, and were determined only by unraveling several cables which had broken over a sheave at one of the tangent points. When this was done, it was discovered that a large number of the outside wires of the strands, and only these, were already broken at the other tangent point, showing that these wires were the first to break.

The writer continued Mr. Stewart's work, and tested 1-in., 6 × 21, 1-in. 6 × 19 Seale, and 1-in. 6 × 19 Warrington, regular lay, non-preformed cables, with just about the same results obtained by the author. However, it was felt that if the wires were nicked by testing them over sheaves, these wires, of course, would be weakened permanently, which would be revealed in a simple tension test. After the several cables had been tested in tension,

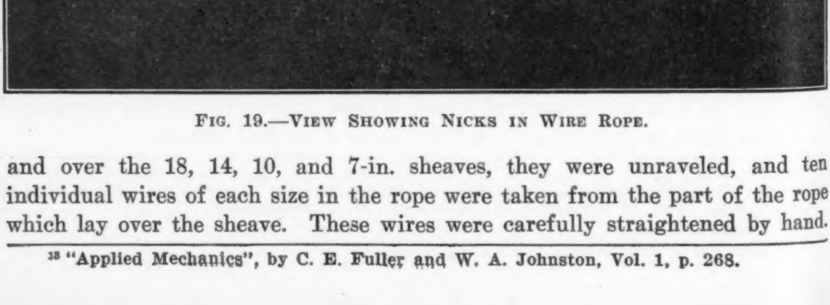

FIG. 19.—VIEW SHOWING NICKS IN WIRE ROPE.

and over the 18, 14, 10, and 7-in. sheaves, they were unraveled, and ten individual wires of each size in the rope were taken from the part of the rope which lay over the sheave. These wires were carefully straightened by hand.

[13] "Applied Mechanics", by C. E. Fuller and W. A. Johnston, Vol. 1, p. 268.

Nicks were noticeable in all the outside wires, and a view of a typical nick is shown in Fig. 19. The reduction in cross-section is clearly shown. On the inside wires, there were no appreciable nicks and only a very small decrease in strength was revealed in the tension tests.

The nicks in the outer wires were larger in the specimens broken over the smaller sheaves, as would be expected, and, consequently, these wires were weaker than those taken from the cables which had been tested over the larger sheaves. This loss in strength of individual wires (diameter, 0.0795 in.) is shown by Curve *A*, Fig. 20. A graph of the loss in strength of the cables (cast-steel, regular lay, non-preformed) over the various sheaves is shown by Curve *B*, Fig. 20, and it is seen that the loss of strength of the cable over the various sheaves, and the loss of strength due to nicking are quite similar. The results shown, are for the 1-in. 6 × 19 Seale, and the results for the other cables were essentially the same. These results show that a large proportion of the loss of strength in cables is due to the nicking effect.

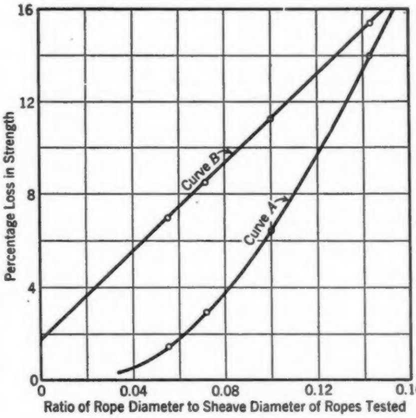

FIG. 20.

The occurrence of these nicks is an added explanation of the failure of cables running over sheaves at fairly low loads. Under the repetition of loads, the wires chafe on each other, enlarging the nicks and reducing the strength until the wire breaks, and this effect is much more pronounced when reverse bends are present.

Stress-strain curves were also drawn of the wires from the broken cables. For some reason, which is most likely the change in cross-section of the individual wires due to abrasion and squeezing in the cable, the stress-strain relationship of the wire seems to change during the testing of the rope. In Fig. 21 are shown the stress-strain curves for a single wire in the rope. The dotted line is obtained from the wires which went into the rope. The full line is obtained from individual wires taken from a broken rope, and this is the curve that should be used in transposing strain data to stress data. If this is done, the load stress curves in the author's Fig. 11, will approach the load-divided-by-net-area line. Similar curves demonstrating this fact for the 1-in., 6 × 19 Seale are shown in Fig. 22. Curve *E* is obtained by using the dotted line in Fig. 21. This curve is seen to be similar to the 1-in. 6 × 7 regular lay, shown in Fig. 11. Curve *C,* in Fig. 22, is obtained by using the solid line in Fig. 21, and, at the breaking load, the stress in the wires is about equal to the average stress. This is what one would expect,

since the plastic flow of the wires above the yield point, re-adjusts the stress throughout the various wires so that they are nearly equally stressed, and at the breaking load they have the same stress.

This condition is contrary to the author's theoretical stress distribution shown in Fig. 15. The writer believes that this distribution is in error for

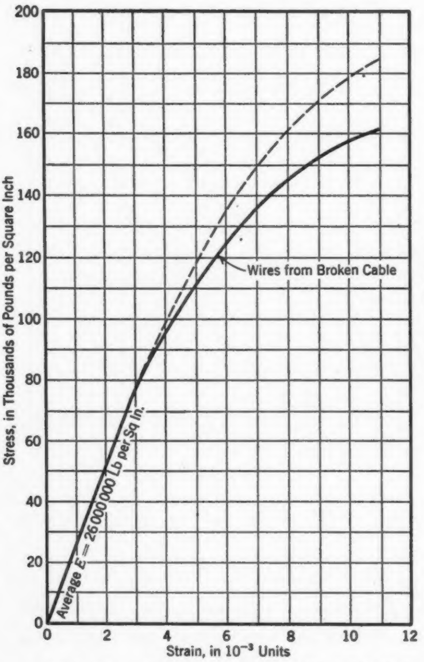

FIG. 21.—AVERAGE STRESS-STRAIN CURVE FOR SINGLE WIRE; 0.0795 INCH IN DIAMETER, CAST-STEEL SPECIMENS 1 to 10.

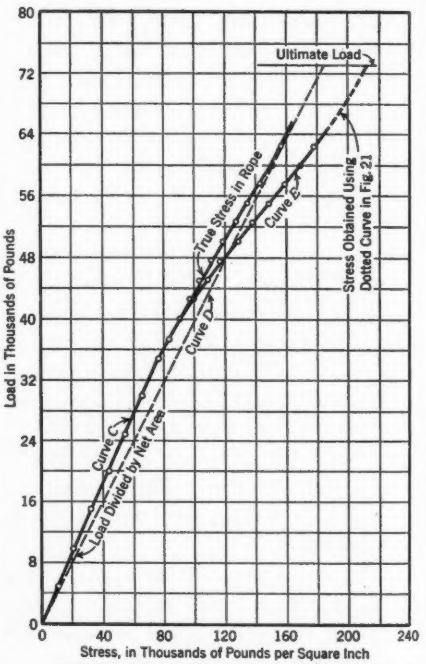

FIG. 22.—LOAD STRESS CURVE FOR TENSION TEST, 1-INCH, 6 x 9 SEALE, CAST, REGULAR, NON-PREFORMED.

several reasons: First, the wrong stress-strain diagram for the single wires was used; second, the instruments used to measure strain, measured it along the chord of the curved wire, and not the actual strain; and, third, the stress in the outer wire is balanced against the assumed stress in the core wire, whereas actually there are enough more outside wires than core wires so that the average stress in the cable is not far from the stress in the outer wires. If these factors were taken into account, the stress in all the wires would be seen to be equal to the average stress as shown in Fig. 22.

A few more remarks on the general behavior of wire ropes may be appropriate. A wire rope may be considered as a single solid bar, or an assembly of individual wires, each acting separately. If thought of as a bar, the usual theories for curved beams would apply. Actually, the behavior of a wire rope is somewhere between these two conditions, the exact degree depending on the relative friction between the wires, the lubrication of the rope, the

resistance of the individual wires to abrasion, and similar factors. The indeterminacy and the variation in these factors would tend to remove wire ropes from the fields of mathematical analysis. All bending stress theories are based on the assumption that the stress of a rope over a sheave varies from a maximum at the outside of the rope to a lesser stress on the sheave; and yet these theories assume that if an individual wire is taken out as a free body for analysis, the stress at the two ends of the wire is the same, and that there are no forces acting along the wire. If a section of any individual wire is taken, it will go from the outside of the rope to the inside. The difference between the two ends of the wire must be due to forces introduced by friction and the bearing of one wire on another, which forces are practically always neglected in analysis, but they cannot be neglected and still give a true picture of what actually occurs in the rope.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

VARIED FLOW IN OPEN CHANNELS OF ADVERSE SLOPE

Discussion

By Messrs. J. C. Stevens, F. T. Mavis, and Hunter Rouse and Merit P. White

J. C. Stevens,[21] M. Am. Soc. C. E. (by letter).[21a]—There seems to be little use for injecting the "varied flow function" into the computations of surface curves. Such functions are not only more laborious to apply, but they are full of inaccuracies as a result of the many basic assumptions and approximations entering into their development

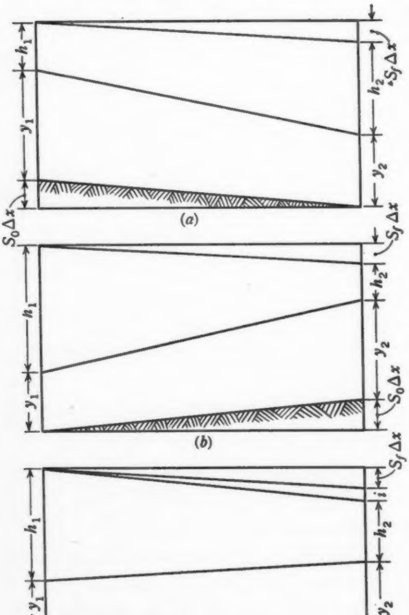

Fig. 7.

Fig. 7 shows the relationships that must obtain between two adjacent sections a distance, Δx, apart. The distance, Δx, is finite, the only limitation being that the surface curve is considered to have a uniform slope in that length. For a sustaining bed slope, Fig. 7(*a*) yields $S_0 \Delta x + y_1 + h_1 = y_2 + h_2 + S_f \Delta x$; whence,

$$\Delta x = \frac{\Delta \epsilon}{S_f - S_0} \quad \ldots\ldots (20)$$

in which $\Delta \epsilon = (y_1 + h_1) - (y_2 + h_2)$—that is, the loss of specific energy in the reach, Δx. It is considered positive downward since there can be no gain in energy with constant flow; S_f is also positive being merely the rate of energy loss by friction.

Note.—The paper by Arthur E. Matzke, Jun. Am. Soc. C. E., was published in February, 1936, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: May, 1936, by Messrs. H. E. von Bergen, W. E. Howland, and Arno T. Lenz.

[21] Cons. Hydr. Engr. (Stevens & Koon), Portland, Ore.

[21a] Received by the Secretary April 20, 1936.

A similar expression from Fig. 7(*b*) for adverse slope, results in,

$$\Delta x = \frac{\Delta \epsilon}{S_f + S_0} \qquad (21)$$

Equations (20) and (21) are perfectly general[22] and apply to any regular channel in which the area is a function of the depth; that is, rectangular, trapezoidal, circular, triangular, parabolic, etc.

If there are energy losses in addition to friction, such as impact or eddy losses, they are deducted from the total energy losses. Thus, the general surface curve formula as indicated in Fig. 7(*c*), may be written,

$$\Delta x = \frac{\Delta \epsilon - i}{S_f \mp S_0} \qquad (22)$$

depending on whether the bed slope is sustaining or adverse. In Equation (22), i = head losses due to impact and eddies.

Whenever velocities are diminishing as in expanding conduits, there is an inherent eddy loss, i, in addition to the normal channel friction. The varied flow functions necessarily assume full recovery of velocity head which is never realized.

TABLE 4.—COMPUTATIONS FOR EXAMPLE 1, APPLYING EQUATION (21)

Depth, y (1)	Area, A (2)	Velocity, V (3)	Velocity head, H (4)	Specific energy, ϵ* (5)	Loss of specific energy, $\Delta \epsilon$ (6)	Wet perimeter, p (7)	Hydraulic radius, R (8)	Kutter's coefficient, c† (9)	Friction slope, S_f, at the section commanded by y (10)	Average friction slope, S'_f (11)	Friction slope, S_f, plus bed slope, S_0, (12)	Length between sections, Δx (13)	Distance, x, from starting point (14)
(*a*) EQUATIONS (20), (21), AND (22) APPLIED TO EXAMPLE 1													
3.0	58.5	8.88	1.226	4.226	.....	25.8	2.26	67	0.00775	....	....	..	0
3.2	63.4	8.20	1.045	4.245	0.019	26.5	2.39	68	0.00610	0.00692	0.00732	2	2
3.4	68.3	7.61	0.900	4.300	0.055	27.2	2.51	68	0.00500	0.00550	0.00590	9	11
3.6	73.4	7.07	0.777	4.379	0.079	28.0	2.62	69	0.00400	0.00450	0.00490	16	27
3.8	78.7	6.60	0.672	4.472	0.093	28.7	2.74	70	0.00324	0.00362	0.00402	23	50
4.0	84.0	6.18	0.594	4.594	0.122	29.4	2.86	70	0.00263	0.00293	0.00333	37	87
4.2	89.5	5.81	0.525	4.725	0.131	30.1	2.97	71	0.00226	0.00245	0.00285	53	130
4.4	95.0	5.47	0.465	4.865	0.140	30.8	3.09	71	0.00193	0.00209	0.00249	56	186
4.6	100.7	5.16	0.414	5.014	0.149	31.6	3.18	72	0.00162	0.00158	0.00198	75	261
4.8	106.6	4.86	0.367	5.167	0.153	32.3	3.30	72	0.00138	0.00150	0.00190	81	342
5.0	112.5	4.61	0.332	5.332	0.165	33.0	3.41	73	0.00117	0.00128	0.00168	98	440
5.5	127.9	4.06	0.256	5.756	0.424	34.8	3.68	74	0.000818	0.000994	0.00139	305	745
6.0	144.0	3.60	0.202	6.202	0.446	36.6	3.94	75	0.000584	0.000701	0.00110	405	1 150
6.5	160.9	3.23	0.162	6.662	0.460	38.4	4.19	76	0.000431	0.000507	0.00091	500	1 650
7.0	178.5	2.91	0.132	7.132	0.470	40.2	4.44	77	0.000320	0.000375	0.00077	610	2 260
(*b*) COMPUTATIONS FOR EXAMPLE 2													
1.5	30.0	36.0	20.15	21.65	.....	23	1.30	123	0.0660	......	......	..	0
2.0	40.0	27.0	11.33	13.33	8.32	24	1.67	127	0.0270	0.0465	0.0469	178	178
2.5	50.0	21.6	7.25	9.75	3.58	25	2.00	130	0.0138	0.0204	0.0208	172	350
3.0	60.0	18.0	5.04	8.04	1.71	26	2.30	133	0.0080	0.0109	0.0113	151	501
3.5	70.0	15.4	3.69	7.19	0.85	27	2.59	135	0.0050	0.0065	0.0069	123	624
4.0	80.0	13.5	2.83	6.83	0.38	28	2.86	137	0.0034	0.0042	0.0046	83	707
4.5	90.0	12.0	2.24	6.74	0.09	29	3.10	139	0.0024	0.0029	0.0033	27	734

* Energy head above the channel bed. † For $n = 0.025$.

Equations (20), (21), and (22) are applied step by step, computing directly without trial—the length, Δx, corresponding to successive increments or

[22] "New Method of Computing Backwater and Drop Down Curves", by Alva G. Husted, *Engineering News-Record*, April 24, 1924, p. 719.

decrements of depth. All that is required is a starting point; the computations may proceed either up stream or down stream. Table 4(*a*) shows the computations for the author's Example 1.

There are no "conveyance" factors involved; nor are normal depths, y_0, normal slopes, S_0, or critical slopes, S_c, required. Before the varied flow functions can be applied, all the various functions of the cross-section for all depths within the range must be computed[23] and curves must be drawn in order to determine the normal depth parameter; the normal and critical slopes, from which to find the constant, β, must also be obtained. The exponent, n, must be found or assumed, and, therefore, a set of computations as complete as those of Table 4(*a*) must be made before one begins to apply the varied flow function—and then one must have the varied flow tables.

All that is required for Table 4(*a*) is a slide-rule and a table of Kutter coefficients. The distance between depths of 3.0 and 5.0 ft by the method herein, outlined, is 440 ft—the author gives 459 ft—a fair agreement.

The curves of Fig. 4 of the paper are certainly in error because the varied flow functions assume complete recovery of velocity head in the expanding water prisms from the gate to the end of the critical depth. Moreover, for the channel and flow given, the critical depth is 4.5 ft instead of 4.4 ft as given by the author.

Table 4(*b*) shows the computations by the method herein outlined. The total length between depths of 1.5 and 4.5 was found to be 734 ft, whereas between depths of 3.0 and 3.5, it is 123 ft. This length, using smaller depth increments, was found to be:

Depth increment, in feet	Distance, in feet, between depths of 3.0 and 3.5 feet
0.5	123
0.2	125
0.1	126

In Example 2 the author gives 160 ft for that distance. In searching for these discrepancies, the writer finds the varied flow functions very sensitive, even to dropping fourth place decimals in interpolation. He also finds[24] that the exponent, n, for this channel is 3.1 for depths less than 4.0 ft, and 2.75 for depths greater than 4 ft. The author used 3.0 throughout.

Table 5 shows the writer's findings for the distances from the gate to the critical depth in the author's Example 2, using the varied flow function. The last value (756 ft) compares favorably enough with the 734 ft of Table 4(*b*).

TABLE 5.—Distances from the Gate to the Critical Depth in Example 2

Exponent, n	Critical depth, y_2, in feet	Distance, in feet, from gate to critical depth
3.0	4.4	821
3.0	4.5	842
3.1 and 2.8	4.5	756

[23] See "Hydraulics of Open Channels", by B. A. Bahkmeteff, M. Am. Soc. C. E., p. 321, for the system of curves required for Example 1.

[24] *Loc. cit.*, p. 318.

The writer found the distance between depths of 3.0 and 3.5 and $n = 3.1$ by differently interpolating the function, but confining the results to three decimal places: $y_1 = 3.0$; $\eta_1 = 0.374$; $B'(\eta_1) = 0.342$ or 0.343; $y_2 = 3.5$; $\eta_2 = 0.405$; and, $B'(\eta_2) = 0.398$ or 0.399; hence, by the author's Equation (13),

$$l_{2\text{-}1} = \frac{8.65}{0.0004} [- (0.405 - 0.347) + 1.17 (0.398 - 0.343)] = 130 \text{ ft}$$

or,

$$l_{2\text{-}1} = \frac{8.65}{0.0004} [- (0.405 - 0.347) + 1.17 (0.399 - 0.342)] = 195 \text{ ft}$$

This shows a variation of 50% in the computed length.

The author gives this length as 160 ft, whereas the writer found 126 ft by his method. All methods (including those of the writer) are in error, however, because complete velocity-head recovery was assumed, and such recovery is impossible of realization.

The simple method outlined herein can readily take into account these eddy losses. They are usually estimated to be some percentage of the negative change in velocity head between adjacent sections. This percentage may be any value from 10 to 100, depending on the character of the channel and the judgment of the engineer; 10% to 20% may be used for artificial channels such as are treated herein, whereas 50% is a fair value for ordinary river channels. The effect of including eddy losses is to shorten the length between given depths.

If the varied flow function is to be used, the tables should be extended to at least four decimal places in order to avoid discrepancies due to interpolation.

F. T. MAVIS,[25] ASSOC. M. AM. SOC. C. E. (by letter).[25a]—Depending upon the reader's point of view this paper may be considered either as a valuable supplement to previous discussions of steady, non-uniform (or varied) flow in open channels, or as an academic treatment of a special case of the broadly general back-water problem. In the first category the paper is so much a part of its antecedent[2] as to be scarcely legible except by frequent reference to it. In his practical examples the author leaves his reader to glean from other sources that, in Fig. 3, Kutter's n is 0.025 and, in Fig. 4, 0.013. Needless reference to authority for definitions of such terms as "the hydraulic exponent, n", and the critical slope, S_c, seems to be unwarranted when these terms are so readily definable. (For a given depth of flow, S_c is the normal (or neutral) slope required to produce a velocity head equal to one-half the hydraulic mean depth; $S_c = \frac{g p}{C^2 t}$, in which g = acceleration of gravity; p = wetted perimeter; C = the Chezy coefficient; and t = width of water

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[25a] Received by the Secretary May 7, 1936.

[2] "Hydraulics of Open Channels", by B. A. Bakhmeteff, M. Am. Soc. C. E., Eng. Societies Monographs, N. Y., McGraw-Hill Co., 1932.

surface; n is twice the slope of the curve on logarithmic paper, showing $K = CA\sqrt{R} = \frac{Q}{\sqrt{S}}$ as ordinates, and the corresponding depth of flow, y, as abscissas.)

In the "Synopsis" the author states that "practical application [of the general equations for varied flow in channels of adverse slope] depends on knowing the numerical values of a function, similar to the 'varied flow function', for canals of sustaining slopes."

It is the writer's opinion that the practical solution of back-water problems cannot be safely discussed in the abstract. In important practical problems one may well be wary of the better-known back-water functions, except as general guide-posts. Ignoring the simple, academic problems, the practical solutions withstand the tests of future observations and the Courts, rather on the strength of reliable basic data and the experience and sound judgment of a competent engineer than on the details of the methods of computation. Among the essentials to the practical solution of back-water problems it may be stated that:

(1) The engineer must be thoroughly acquainted with the conduit. Given adequate profiles, cross-sections, and field data, he must draw on a rich background of experience in the assignment of roughness factors and effective flow sections in curved and irregular conduits.

(2) The relation between flow and corresponding head losses must be assumed. In the simpler cases this is usually expressed as a formula for the mean velocity of flow in terms of the roughness factor, hydraulic radius, and the friction slope.

(3) Obviously, the computer must be well-grounded in the fundamentals of hydro-mechanics—work and energy (Bernoulli's theorem), and impulse and momentum—and in their application. It is desirable that he possess an aptitude for simplifying and idealizing a given problem for purposes of computation without introducing significant or uncertain errors.

In practice, those who must solve back-water problems of importance are likely to develop their own methods of analysis—based generally on the equivalent of the author's Equation (2) and applied to the effective conduit.

In back-water problems of rivers the distances between valley cross-sections are usually fixed by the surveys. Therefore, explicit relations for length of reach, similar to Equation (7), may not be as convenient as a more elementary form of equation, based on steady flow within a short reach, solved by trial.

Fundamentally, there is no difference between the author's method of computing back-water curves by the so-called "varied flow functions" and methods of successive approximation applied to short reaches of a stream. For a short reach, a regular channel, and steady flow, Fig. 1 and Equation (2) apply equally well to all methods.

In the method of successive approximations each reach presents a definite problem of determining for a given discharge the relation between water-

surface elevation and slope at the mean section. Furthermore, the water-surface elevations at the common sections of adjacent reaches must coincide. A consistent water profile is obtained by successive trial computations. The difference in water-surface elevations at two sections is found simply by the numerical addition of the products of each intervening length of reach and its corresponding surface slope.

In the author's method the problem which involves definite physical factors, such as discharge, in cubic feet per second, elevations (or depths), in feet, and slopes of water surfaces, is early obscured by the introduction of the dimensionless ratios, $\eta = \frac{y}{y_0}$, $\beta = \frac{S_0}{S_c}$, and $n = 2\ \frac{y}{K}\left(\frac{dK}{dy}\right)$. Each of these ratios is a function of y, the depth of flow in the conduit, although there may be no significant error introduced by assuming β and n to be constant for purposes of integration. Although integration is often a convenient method of finding the sum of a column of figures that represent some regular, analytical (or graphical) array of numbers, the integration leading to the B' (η)-functions replaces, in part, the numerical addition of products of each length of reach, and the corresponding surface slope aforementioned—nothing more.

For the solution of back-water problems in regular, artificial conduits, such as canals, sewers, etc., it may be that the so-called "varied flow functions" may effect a saving in time over the more elementary method of successive approximations. However, for practical river problems in which back-water curves are important, the author's method seems to be distinctly inferior to the method of successive approximations in which there is no necessity for overlooking such factors as roughness and effective sections, and obscuring such physical quantities as elevation, slope, and discharge, by introducing dimensionless ratios.

HUNTER ROUSE,[26] ASSOC. M. AM. SOC. C. E., AND MERIT P. WHITE,[26] JUN. AM. SOC. C. E. (by letter).[26a]—An effort such as this, to make available to hydraulic engineers additional means of simplifying open-channel design, should be welcomed by the hydraulic world. Although the subject of sustaining slopes has been covered amply elsewhere, this is the first successful attempt to provide both an outline of procedure and tabular values for the solution of problems of adverse slope. The labor of determining the numerous values of the B' (η)-function is not to be under-estimated, although the many hours given by the author to this tedious computation have been well spent.

It seems to the writers that the author's method of arriving at what should be a general equation of varied flow does not lend full clarity to the physical principles involved, for the procedure adopted with respect to signs tends to conceal the basic nature of the problem rather than to explain it. Indeed, the use of an upward—and hence negative—slope at times as a positive value is often confusing.

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[26] With Soil Conservation Service, California Inst. of Technology, Pasadena, Calif.

[26a] Received by the Secretary May 22, 1936.

Mr. Matzke has designated a downward slope as positive and an upward slope as negative; to be consistent, this notation must apply as well to the slope of the water surface and of the energy line as to the slope of the channel bottom. The symbol for the slope of the water surface, S_w, for instance, therefore, denotes either a downward slope or an upward slope, depending upon whether its value is greater or less than zero.[28b] Referring to Fig. 8, the elevation of the free surface at any section, regardless of slope, is equal to the depth, y, plus the elevation of the channel bottom, h_0. Letting H_p represent the total potential head, or the vertical distance from the assumed geodetic datum to the water surface:

$$H_p = y + h_0 \qquad (23)$$

The rate of change of this surface elevation in the positive direction of flow is then simply the derivative of the potential head with respect to x:

$$\frac{dH_p}{dx} = \frac{dy}{dx} + \frac{dh_0}{dx} \qquad (24)$$

Since a downward slope is positive (that is, greater than zero), and signifies a negative rate of change of elevation,

$$\frac{dH_p}{dx} = - S_w = \frac{dy}{dx} - S_0 \qquad (25)$$

Equation (25) is offered in preference to Equation (1), for it is mathematically correct for both positive and negative slopes of the water surface and for both sustaining and adverse slopes of the channel bottom.

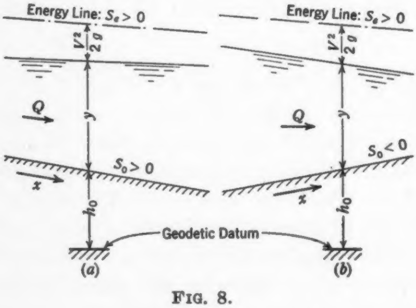

FIG. 8.

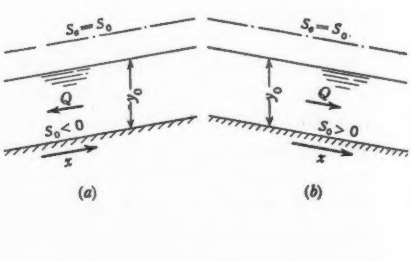

FIG. 9.

Proceeding in a similar fashion, the total head, H, above the arbitrary geodetic datum at any section is equal to the sum of the velocity head and the potential head (refer to Fig. 8):

$$H = \frac{V^2}{2\,g} + y + h_0 \qquad (26)$$

[28b] For *Transactions*, the symbol for slope, s, will be changed to read S throughout.

The rate of change of total head in the positive direction of flow is then simply the derivation of this expression, with respect to x:

$$\frac{dH}{dx} = \frac{d}{dx}\left(\frac{V^2}{2\,g}\right) + \frac{dy}{dx} + \frac{dh_o}{dx} \qquad (27)$$

Since the positive rate of change of total head is equal to the negative rate of loss of total head, $\frac{dH}{dx} = - \frac{d\,e_r}{dx}$, the latter derivative being used in Equation (2); and since a positive rate of loss corresponds to a positive or downward slope,

$$\frac{dH}{dx} = - S_e = - \frac{d\,e_r}{dx} = \frac{d}{dx}\left(\frac{V^2}{2\,g}\right) + \frac{dy}{dx} - S_o \qquad (28)$$

Equation (28) is offered to replace Equation (2); it is quite as general as Equation (25).

The Chezy equation,

$$Q = A\;C\,\sqrt{R\;S_e} \qquad (29)$$

expresses the rate of discharge, Q, in terms of the cross-sectional area of flow, A; a coefficient, C; the hydraulic radius, R; and the slope of the energy line, S_e. The first three terms, A, C, and R, depend only upon the cross-sectional shape and roughness of the channel and the variable depth of flow, and may all be determined as functions of y for a given channel. As shown by Bakhmeteff[2], the product, $A\,C\,\sqrt{R}$, as a function of y may conveniently be given the symbol, K, to denote the "conveyance" of the channel; hence,

$$Q = K\,\sqrt{S_e} \qquad (30)$$

If the discharge and the conveyance are such that for the given bottom slope, S_o, the depth of flow is uniform over the entire channel, then $y = y_o$ and the slope of the energy line, S_e, is equal to both the surface slope, S_w, and the bottom slope, S_o. For such conditions,

$$Q = K_o\,\sqrt{S_o} \qquad (31)$$

The term, K_o^2, is used as a convenient relative parameter for flow on sustaining slopes, and has been adopted by the author for adverse slopes as well. Since uniform flow on adverse slopes is an artificial condition, such usage may well bear further analysis.

Although S_w and S_o may be either positive or negative with flow in the positive x-direction, a negative value of S_e can only signify flow in the negative x-direction, for the energy line must invariably slope downward in the direction of flow. Hence, the equation of uniform flow,

$$Q = A\;C\,\sqrt{R}\,\sqrt{S_o} = K_o\,\sqrt{S_o} \qquad (32)$$

[2] "Hydraulics of Open Channels", Eng. Societies Monograph, McGraw-Hill Co., 1932.

when used as a flow parameter, must be such that for negative values of S_0 the discharge will also be negative (that is, it will still be in the direction of decreasing energy and downward bottom slope (see Fig. 9)). Thus, Q^2 and $K_0^2 S_0$ are invariably positive, regardless of whether the numerical value of S_0 is positive or negative.

A plot of $S_0 = \frac{Q^2}{K_0^2} = \frac{\text{constant}}{K_0^2}$, shown schematically in Fig. 10, indicates clearly the fact that for negative values of S_0 the conveyance, K_0, must be

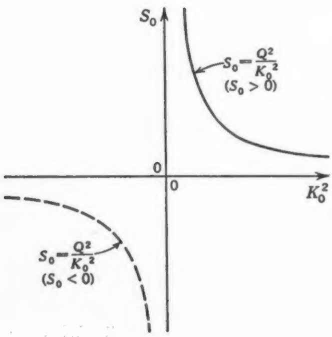

FIG. 10.

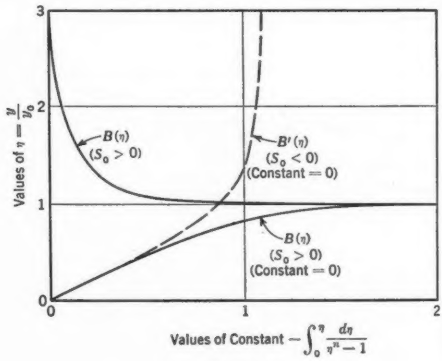

FIG. 11.

imaginary, since its square is negative. This may be true only if the coefficient, C, becomes imaginary for all negative values of S_0, since A and R are invariably positive. In the relationship, $Q^2 = K_0^2 S_0$, used by the author following the procedure outlined by Bakhmeteff[3], the imaginary root, $i = \sqrt{-1}$, contained in K_0 for negative values of S_0, therefore, cannot be neglected.

Such discontinuity in the conveyance function will be more clearly understood if one recalls that as the slope of a channel approaches zero, the discharge thereby remaining constant, the uniform depth must approach infinity in order that the slope of the energy line may also approach zero. If the slope of the bottom continues to decrease, once it is less than zero uniform conditions are quite impossible unless the flow reverses in direction. Continued use of the uniform flow relationship of Equation (32) as a parameter for varied flow in the positive x-direction naturally requires that this equation be mathematically consistent.

The following substitutions are now in order, after the method taken by the author from the cited text[3]:

$$S_e = \frac{Q^2}{K^2} = S_0 \frac{K_0^2}{K^2} \qquad (33)$$

$$\frac{d}{dx}\left(\frac{V^2}{2\,g}\right) = -\frac{Q^2}{g}\,\frac{b}{A^3}\,\frac{dy}{dx} = -\frac{S_0\,K_0^2}{g}\,\frac{g}{S_c\,K^2}\,\frac{dy}{dx} = -\frac{S_0}{S_c}\,\frac{K_0^2}{K^2}\,\frac{dy}{dx} \qquad (34)$$

and,

$$\beta = \frac{S_0}{S_c} \dots\dots\dots\dots\dots\dots\dots\dots\dots (35)$$

Introducing Equations (33), (34), and (35) in Equation (28),

$$- S_0 \frac{K_0^2}{K^2} = - \beta \frac{K_0^2}{K^2} \frac{dy}{dx} + \frac{dy}{dx} - S_0 \dots\dots\dots\dots (36)$$

and, since K^2 varies as y^n:

$$\frac{dy}{dx} = S_0 \frac{1 - \dfrac{K_0^2}{K^2}}{1 - \beta \dfrac{K_0^2}{K^2}} = S_0 \frac{1 - \left(\dfrac{y_0}{y}\right)^n}{1 - \beta \left(\dfrac{y_0}{y}\right)^n} \dots\dots\dots\dots (37)$$

Equation (37) is quite general, applying to positive and negative slopes of the water surface and sustaining and adverse slopes of the channel bottom; as such it is offered as an improvement on Equation (4). One need only remember that when S_0 is negative, K_0^2, and hence, y_0^n, must have negative numerical values owing to the square of the imaginary *i*. That β will, of itself, become negative is quite evident, since it then represents the quotient of two slopes of unlike sign, S_c always being positive.

From the relationships $\eta = \dfrac{y}{y_0}$ and $dy = y_0 \, d\eta$:

$$\frac{y_0}{S_0} \frac{d\eta}{dx} = \frac{1 - \dfrac{1}{\eta^n}}{1 - \beta \dfrac{1}{\eta^n}} = \frac{\eta^n - 1}{\eta^n - \beta} \dots\dots\dots\dots\dots (38)$$

which becomes, through division,

$$\frac{S_0}{y_0} dx = \frac{\eta^n - \beta}{\eta^n - 1} d\eta = d\eta + (1 - \beta) \frac{d\eta}{\eta^n - 1} \dots\dots\dots (39)$$

Equation (39) is offered to replace Equation (5). The term, η^n, now becomes negative with negative values of S_0, again owing to the imaginary value of C.

Integration of Equation (39) between the limits, x_1 and x_2, and η_1 and η_2, will be as follows:

$$\frac{S_0}{y_0} (x_2 - x_1) = \frac{S_0}{y_0} l_{2\text{-}1} = \eta_2 - \eta_1 + \int_{\eta_1}^{\eta_2} (1 - \beta) \frac{d\eta}{\eta^n - 1} \dots\dots (40)$$

Equation (40) is completely general, and as such is to be recommended instead of Equation (6). For positive values of S_0, after Bakhmeteff[3].

$$\int_0^{\eta} \frac{d\eta}{\eta^n - 1} = \text{constant} - B(\eta) \dots\dots\dots\dots\dots\dots (41)$$

When $\eta < 1$, the constant has been taken as zero in computing the values of $B(\eta)$ given by Bakhmeteff[2]. Equation (40) then becomes, for $S_o > 0$,

$$l_{2-1} = \frac{y_o}{S_o}\left[\eta_2 - \eta_1 - (1-\beta)\left(B(\eta_2) - B(\eta_1)\right)\right] \quad \text{.........(42)}$$

For negative values of S_o and hence of η^n, following the same designation as to sign,

$$\int_0^\eta \frac{d\eta}{\eta^n - 1} = -B'(\eta) \quad \text{..................(43)}$$

in which the numerical values of the function, $B'(\eta)$, are correctly given in Table 2. Thus, when $S_o < 0$,

$$l_{2-1} = \frac{y_o}{S_o}\left[\eta_2 - \eta_1 - (1-\beta)\left(B'(\eta_2) - B'(\eta_1)\right)\right] \quad \text{........(44)}$$

which is fully commensurate with Equation (42), and as such is offered in preference to Equation (7). The form of the function, $-\int_0^\eta \frac{d\eta}{\eta^n - 1}$, for values of S_o and η^n greater and less than zero is indicated in Fig. 11.

The writers believe that adherence to the foregoing rigorous and general development will avoid confusion in the future treatment of varied flow; this development will also explain the apparent dissimilarity in the equations for adverse and sustaining slopes in the paper.

In closing, two essential points already discussed at length by Professor Bakhmeteff[2] might well be emphasized. First, the development of the varied flow relationships is based entirely upon the assumption of motion in which the curvature of the stream lines is not appreciable, and in which the pressure distribution, therefore, is hydrostatic. The surface curve is thus a function of varying resistance and is independent of the dynamic effects of rapid acceleration or deceleration. In the neighborhood of the critical depth, however, curvature actually becomes of major importance, and only because of the great length of channel involved in the resistance curve can this phenomenon of local transition be disregarded.[27] Second, success in applying these relationships in the accurate design of channels depends entirely upon the proper determination of the Chezy coefficient, C. At present, this must depend upon one or another of the several empirical relationships for C, not one of which can stand a rational analysis. Only when resistance in open channels is as well understood as that in circular pipes can the treatment of open-channel flow be improved by more sound physical relationships; however, although pipe resistance depends largely upon wall roughness, the open channel introduces as further variables the cross-sectional form and the non-uniformity of longitudinal profile, both of which can influence C to an appreciable degree. This is probably the most fertile field of present-day hydraulic research—surely the most imperative.

[27] See "Discharge Characteristics of the Free Overfall", by Hunter Rouse, *Civil Engineering*, April, 1936, p. 257.

AMERICAN SOCIETY OF CIVIL ENGINEERS

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DISCUSSIONS

SURFACE AND SUB-SURFACE INVESTIGATIONS, QUABBIN DAMS AND AQUEDUCT

A SYMPOSIUM

Discussion

BY MESSRS. BAYARD F. SNOW, AND OLE SINGSTAD

BAYARD F. SNOW,[16] M. AM. SOC. C. E. (by letter)[16a].—The operations preliminary to the construction of the Quabbin Dam and Dike afforded an excellent opportunity for the collection and study of many data relating to the exhaustive pumping of ground-water. Such data are of intense interest to many engineers whose work concerns the flow of ground-water, and Mr. Dore deserves much credit for the manner in which he has collected and presented the facts. Undoubtedly, the problem of estimating the expected leakage through the earth over-burden under a dam is nearer a rational and reasonable solution as a result of this paper. There is a question, however, as to whether the data and study are of most value in connection with estimating the flow under proposed dams. It is true that, in the case of the Quabbin works, the bore-holes were made and some of the pumping was done prior to the actual beginning of core-wall construction even if such holes may not have preceded the decision to construct such a water-stop. There is no doubt that they assisted greatly in the construction of the core-wall. However, much of the information was gathered during the sinking of the caissons and, therefore, can have had no part in reaching the decision that caissons were necessary.

It would be of considerable value to the Engineering Profession if Mr. Dore would present a modification of his study of these basic records to assist in answering a series of questions frequently met in the study of the flow of ground-waters; for example:

(*a*) What is the probable safe yield and the optimum spacing of wells under given geologic and topographic conditions?

NOTE.—This Symposium by Frank E. Winsor, M. Am. Soc. C. E., and Stanley M. Dore and Frank E. Fahlquist, Assoc. Members, Am. Soc. C. E., was published in March, 1936, *Proceedings*. This discussion is printed in *Proceedings* in order that the views expressed may be brought before all members for further discussion of the Symposium.

[16] Cons. Civ. and San. Engr.; Director, X. Henry Goodnough, Inc., Boston, Mass.

[16a] Received by the Secretary May 15, 1936.

(*b*) What portion of the pumpage is depletion of storage, what part is infiltration from within the area of the circle of influence, and what part a ground-water flow from outside that circle? and,

(*c*) With soil of a given porosity, what part of those pores is filled with water and how much of that water can be drawn as the water level is lowered by pumping?

Fig. 10 indicates that there was no change in water level at points ½ mile up stream and down stream after eighty weeks of pumping, although at the caisson it had dropped 100 ft below its original level. Pumping rates for a year had been about 3 mgd, after which they had increased so as to average slightly more than 5 mgd. One may assume that, at a point ½ mile above the dam, there was no change in the direction and velocity of the flow of ground-water and surface water as the result of the pumping. The easiest hydraulic channel for water precipitated on the uplands above this point was by surface and underground courses to the river, and this condition was unchanged by the pumping. It seems obvious, therefore, that in such materials the watershed from which a ground-water supply could be taken by wells near the site of the dam was less than a mile in diameter and that the safe yield would be not more than one-tenth the rate of pumping actually used in dewatering the soil, and even that only by excessive draw-down. It is probable that with a slight change in average size of grain, the radius of influence would be extended greatly. The real question is whether a circle of influence or area contributing to the yield of a ground-water supply can be predicted by analysis of soil samples. If Mr. Dore can utilize his data to this end and give the Engineering Profession a workable means of determining yield of wells under given geologic conditions, a noteworthy contribution to water supply engineering will have been made.

The determination of permeability by borings is certain to be, in a large measure, a matter of luck. Mr. Dore calls attention to the random nature of the deposits and refers to the connection of the coarser deposits by links of relatively small cross-section, but which serve very effectively in conducting the flow from one coarse lens to another. In seeking a suitable site for an additional ground-water supply, the writer has had the experience of finding what appeared to be a pocket of coarse material completely surrounded by a very fine sand. As far as borings could reveal any information, an acre or two of coarse sand was available, which would serve as a collecting and storing reservoir, surrounded by material through which water would flow at very slow rates. The area between the old well field and the new site was apparently an almost impervious barrier. On a pumping test, however, it was shown that hydraulic communication between the two areas was free and that each well field responded promptly to pumping from the other. It would appear, therefore, that the zone of influence was much larger than would have been indicated by computations based on examination of boring samples.

In this area the writer saw an interesting demonstration of the transmission of pressure through ground-water. On one well a stroke of a hand-pump

caused an immediate temporary drop of an inch or more in the water level at the other well, 80 ft, or more, away. This calls attention to the fact that ground-water flow is affected not only by the permeability of the water-bearing strata, but also by the resistance to transmission of air through the overlying material. If the water-bearing material adjacent to the first well had been freely open to atmospheric pressure, there would have been little discernible effect of the vacuum a few feet away from the well. Many test wells in the writer's experience have disclosed nothing which would indicate a potential water supply, all the material above ledge being nearly impervious. Frequently, however, a thin stratum of coarse material lies at the rock surface and serves to form a channel through which the water flows freely and perhaps, by reason of the nature of the overlying material, transmits a differential pressure for considerable distances to points where the over-burden is less in amount or more pervious. When deposits of coarser material are connected by such devious channels the total effective water-shed to a well may be much greater than study of the most carefully selected boring samples would indicate was possible

The ratio of permeability between the material of such a very thin coarse stratum and the overlying fine deposit may be 100 to 1 so that it would appear to be important to measure the water-bearing stratum with extreme care. The ordinary methods of wash-boring and sampling do not permit of such refinement, even if by luck the boring penetrates a representative section of such a stratum. This being so, although the writer agrees that water tends to open passages of flow between pockets of coarser materials and feels that such passages may have a carrying capacity equal to that of the deposits of coarse sand they connect, nevertheless, the determination of the average permeability of a cross-section by boring samples is subject to great errors, especially where the borings miss the larger masses of relatively coarse material and may or may not penetrate a typical cross-section of the connecting channel. Is it not more simple and more accurate to utilize the test borings that would be necessary for soil sampling, pumping from some and observing the slope of the water surface at others, rather than to depend upon samples obtained by more or less blind groping?

OLE SINGSTAD,[17] M. AM. SOC. C. E. (by letter).[17a]—Being especially well prepared the valuable paper by Mr. Fahlquist presents an excellent picture of the geologic features of an extensive project and their relation to its construction operations and cost. The most impressive aspect of the paper is the manner in which the geologic features have been taken into consideration in the planning of the tunnel project. It illustrates the importance and value of the co-operation between the engineer and the geologist in the proper planning of an underground project of this magnitude. It is quite evident that this thorough and scientific consideration of the geological features in the planning was in large measure responsible for the successful construction operations carried on without untoward happenings. This complete preliminary informa

[17] Chf. Engr. New York City Tunnel Authority, New York, N. Y.

[17a] Received by the Secretary April 3, 1936.

tion undoubtedly removed from the project much of the uncertainty attached to the bidding on underground projects where the preliminary exploratory work is less thorough, resulting in economy to the owner and a greater degree of certainty that the contractor can complete the work within the bid price and the stipulated time.

The paper also presents in convenient tabular form much valuable information as to cost and progress, and shows the variation of these items in relation to the geologic conditions. It should be of great value to engineers dealing with underground construction work, in stimulating more attention to the geologic features and more thorough exploratory work than has sometimes been the case in the past.

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DISCUSSIONS

ADMINISTRATIVE CONTROL OF UNDERGROUND WATER: PHYSICAL AND LEGAL ASPECTS

Discussion

BY MESSRS. JOSEPH JACOBS, AND W. D. FAUCETTE AND J. E. WILLOUGHBY

JOSEPH JACOBS,[7] M. AM. SOC. C. E. (by letter).[7a]—This exceptionally able paper deals with the subject of the administrative control of underground water—a consideration of steadily increasing importance in the economic development, not only of the semi-arid sections of the West, but of the entire country. In his reference to irrigation in Oregon and Washington, the author inadvertantly states that "west of the Cascade Mountains irrigation is not practiced, but east of these mountains both States are arid." The writer desires to offer a correction of that statement.

As to the sections east of the Cascades it would be more accurate to state that they would, in general, be classed as semi-arid although there are considerable areas along the eastern border of Washington, and to some extent in Northeastern Oregon, that have sufficient rainfall for successful agriculture without irrigation. The status of irrigation west of the Cascades is portrayed in the following quotation from a paper prepared by the writer for the Pacific Northwest Regional Planning Commission and which appeared, in abstract, in a report published in 1936 by the National Resources Committee:

"There has, as yet, been but limited irrigation development west of the Cascades. A relatively high annual precipitation, and a relatively high humidity, even in summer, have served to discourage such development. However, despite the rather substantial annual precipitation, its 3 inches or less of summer rainfall, when water is most needed by the crops, is less than that of many irrigated sections on the east side. The cost of west-side irrigation should, in general, be much less than that of the east side due to the shorter irrigation season, the less amount of water required, and the more readily available water supplies. Unquestionably, the agricultural output

NOTE.—The paper by Harold Conkling, M. Am. Soc. C. E., was published in April, 1936, *Proceedings*. This discussion is printed in *Proceedings* in order that the views expressed may be brought before all members for further discussion of the paper.

[7] Cons. Engr., Seattle, Wash.

[7a] Received by the Secretary May 20, 1936.

would be increased by such irrigation and unquestionably, too, the expense involved would, in many instances, be fully justified economically. About 12 000 acres are now irrigated in western Washington and 65 000 acres in western Oregon, of which about 10 000 acres are in the Willamette Valley. A recent study made by the Oregon State Planning Board, reports 1 140 000 acres of irrigable land in that valley. A more detailed investigation of the valley is now being made by the Corps of Engineers, United States Army. It is easily conceivable that west-side irrigation may ultimately amount to as much as 1 500 000 acres."

W. D. FAUCETTE[8] AND J. E. WILLOUGHBY,[9] MEMBERS, AM. SOC. C. E. (by letter).[9a]—In view of the prevailing lack of administrative control of the ground-waters and the reckless destruction of those waters by interests having other objects in view, this paper is particularly timely. There is now (1936) a determined effort, well-financed, to obtain the excavation of a sea-level ship canal by the Federal Government across the underground flow in the peninsula of Florida, which has aroused the apprehension of the many people whose farming and fruit-growing industry is dependent on that flow. The physical conditions in Florida make a résumé thereof an item of interest in connection with Mr. Conkling's paper.

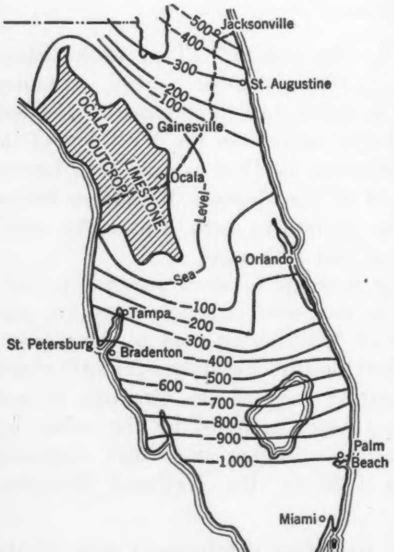

FIG. 1.—STRUCTURAL MAP WITH CONTOUR DRAWN ON TOP OF THE OCALA LIMESTONE IN FLORIDA.

The author discusses conditions that obtain in the arid West; in Florida, the rainfall is generous, and the geological relationships are different from any mentioned in his paper. Artesian water in Florida is not found in structural basins, but it moves outward and downward in a great dome of Ocala limestone. This limestone (which is a porous, soft, cavernous material through which ground-water moves freely) lies at or near the surface in the area from about Webster and Bushnell on the south, to Silver Springs on the east, and Gainesville, High Springs, and Mayo on the north, and extends to the Gulf of Mexico (see Fig. 1)[10]. The flanks of this limestone outcrop are covered with younger and often more impervious formations. The Ocala limestone outcrop area forms a great catchment basin. Copious rainfall thereon percolating outward and downward through this formation which underlies the entire State of Florida and is the most important water-bearing formation

[8] Chf. Engr., Seaboard A. L. Ry. System, Norfolk, Va.

[9] Chf. Engr., A. C. L. R. R., Wilmington, N. C.

[9a] Received by the Secretary May 27, 1936.

[10] Twenty-First and Twenty-Second Annual Repts., Florida Geological Survey, Fig. 8, p. 50.

in the State, is the source of a large percentage of the ground-water supply available through deep wells and springs in the Peninsula.

A most important and pertinent question raised by the advocates for the conservation of the ground-waters of Florida is the effect of a 200-mile ditch cut 30 ft below sea level, across the catchment area that supplies the greater part of the artesian water of Florida. Would such a ditch so intercept, contaminate, and pollute the ground-waters of the Florida Peninsula as to destroy much of the beauty and charm of the great winter playground of the Nation, and, at the same time, ruin its groves, farms, and its market gardens? Would the canal bring about a situation whereby the residents of Florida could no longer depend upon ground-water supplies for industrial, municipal, agricultural, and domestic uses?

For several years, in co-operation with the U. S. Geological Survey, the Florida State Geological Survey has been making a study of the ground-waters of the State. Realizing the danger of promiscuous borings for artesian water and of contamination by drainage wells, the State Geologist has advocated a very careful supervision and conservation of ground-water supplies, and laws have been passed to that end. In spite of these recommendations, certain engineers and geologists have made superficial investigations and recommended the building of the canal, without giving due consideration to its effect upon ground-waters.

To the present time the weight of scientific evidence as to the effect of a canal on the geological problems involved, has been against, rather than for, the building of the canal. The geologists who have reported favorably on the canal admit the possibility of grave danger to such scenic features as Silver Springs and Blue Springs, but state that this danger may be minimized or entirely prevented by proper engineering precautions. However, lowering the water-table to 40 ft below the present level of Silver Springs and Blue Springs, and the lower courses of the Withlacoochee and Ocklawaha Rivers, will undoubtedly destroy these springs. That is what an effective drainage ditch will always do—drain. It is difficult to imagine any engineering precaution that can be taken to prevent it. The author points out that springs in the vicinity of Roswell, N. Mex., no longer exist, because the water formerly issuing from them has been diverted, and the area of artesian flow within that region has shrunk from 663 sq miles to 425 sq miles, because of depressed piezometric head.

In regard to the effect of salting artesian wells, the geologists favoring the canal frankly admit that they do not know what will be the result, if the canal is constructed. However, Dr. Herman Gunter, Florida State Geologist, specifically states that such salting is greatly to be feared, especially in that area where highly mineralized water is present at moderate depths. Any lowering of head due to excessive drainage in the Ocala catchment area may cause further encroachment of salting in those areas where there already exists a delicate balance between fresh water and salt water. This condition is already faced by some vegetable growers in the Sanford Section. Less than normal rainfall on the catchment basin over a period of several years has reduced the quantity of ground-water available in the Ocala formation,

and this, together with the attendant necessity of heavy draft on flowing artesian wells for the irrigation of crops, has resulted in some wells in the Sanford District ceasing to flow, and the waters in others showing high salinity. The construction of the proposed canal could accentuate and make permanent these conditions, not only in the Sanford area but also in other important vegetable and fruit-producing districts of the State. Growers in the Sanford District, having experienced the disastrous effects on their crops of depleted ground-water supplies, have arisen *en masse* protesting the construction of the canal, fearing that it would completely destroy their livelihood. It is also true in certain areas, such as Sanford, that any diminution of head would necessitate the pumping of wells where they now flow freely. This would greatly increase the cost to the growers of water used for irrigation.

Geologists and engineers who have favored building the canal have definitely stated that the pursuit of agriculture and the growth of vegetation will not be affected; that the progress of agriculture and the growth of vegetation in general are only remotely related to the water-table. These are strong statements when it is remembered that three-fifths of the rainfall of Florida is in June, July, August, and September (which is the non-growing season, as Florida farmers produce their crops during the winter months for the Northern markets), and evaporation is so great that probably less than 20 of the 50 in. that fall as rain sinks into the soil. The control of water through irrigation and drainage is rapidly becoming a limiting factor in successful crop production in all areas of the United States. Areas in Florida that are not irrigated, and where the water-table is 60 to 100 ft below the surface, are adapted, only under exceptional conditions, to a highly developed agriculture. Florida growers are beginning to realize the necessity for more precise control of water in crop production if the State is to mtaintain its horticultural prominence. In that State, with the exception of the Everglades, irrigation is mostly from the underground water supply. Vast producing districts have been established in many parts of the Peninsula where perishable crops are grown under intensive methods based on irrigation from deep wells. This shows that Florida farmers realize that they cannot depend entirely on rainfall for the production of their crops. Like many areas in the West, noted by the author, Florida is using more and more artesian water for irrigation, and for municipal, industrial, and domestic purposes; and there is, therefore, a constantly increasing drain on underground supplies.

The statement has been made that although the water-table at the edge of the canal would be lowered to sea level, normal water-table slopes would be approached 10 to 15 miles from the canal. Unfortunately, the areas used in calculating these slopes were not taken in the Ocala limestone, but were in the Tampa and Hawthorne formations, which are younger and more impervious. As the Ocala material has a mean porosity of 30.6%, and as the canal would cross "the ramifying system of ground-water drains which has been developing over a period of millions of years", discharge from this area would be very rapid and the water-table slopes in such a formation much more gentle, than those used in the calculations previously cited.

Over a large area in Central Florida the water-table stands between 40 and 50 ft above sea level. With more complete drainage established by cutting into the Ocala limestone, the water-table will fall to sea level over approximately the same area with the following results: (1) Decrease of head will cause salt water to rise in artesian wells approximately 40 ft for every foot the head is reduced; (2) lowering the water-table will remove supports from cavern roofs and walls, and there will be a progressive formation of sink holes from the canal, both to the north and south; and (3), the areas of high water-tables (the piezometric surface), both north and south of the canal, will be lowered, whereas high water-tables have a beneficial effect on temperature during cold waves.

Amplifying Result (2), perched lakes and swamps will be drained. It is interesting to note that many of the lakes of Florida may be of this character. A notable example of a lake of this type existed at Payne's Prairie, near Gainesville, Fla. Payne's Prarie has an area of 18 to 20 sq miles. In the summer of 1790, William Bartram visited this section and found a vast savanna that afforded pasturage to large herds of horses and cattle belonging to the Alachua tribe of Indians. When visited by James Pearce, in 1824, this basin was still dry land. About 1871, Alachua sink became clogged, and the body of water thus formed was known as Alachua Lake, which is reported to have been navigable for small steamers. This lake continued until the summer of 1891, when it was drained through a sink. Since that time the lake bed has remained dry land, with the exception of temporary overflows, and is known as Payne's Prairie, one of the best cattle-grazing areas in the State.

As the author points out, the use of ground-water in the United States as a whole is constantly increasing, and it is important that this valuable natural resource be conserved. Ground-water is the very life blood of Florida; and in view of this fact one can readily understand the concern which many localities in that State, south of the route of the proposed canal, have shown with regard to the conservation of their most important natural resource. It is the hope of the writers that the present agitation in Florida will lead to more effective administrative control, and to the development in the mind of engineers and of the public of a greater sense of responsibility to the future of preserving the ground-water resources throughout the Nation.

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DISCUSSIONS

FLOOD PROTECTION DATA

PROGRESS REPORT OF THE COMMITTEE

Discussion

BY C. R. PETTIS, M. AM. SOC. C. E.

C. R. PETTIS,[5] M. AM. SOC. C. E. (by letter).[5a]—The five recommendations of the Committee are thoroughly sound, and represent studies which are necessary for a proper consideration of various flood problems. There is no sharp line of demarcation between cloudbursts and storms of other types; but for certain practical reasons it is desirable to make a study of cloudbursts, considered as an independent type.

It is suggested that Man's occupancy of the land has introduced certain problems, and in this particular connection erosion is probably of more importance than the effect on flood peaks. Any studies made under Recommendation (4) should give careful consideration to erosion. Referring to Recommendation (5), the writer is of the opinion that benefits can not be properly evaluated without giving consideration to damages, and the probable frequency of their occurrence.

The recommendations should have included a study of flood probabilities, which is a primary element that must be considered in connection with flood-control problems. The probable 100-yr flood, with which most engineers are familiar, is a convenient basic unit to use in connection with probability studies; one primary purpose of the proposed study should be the assembly of data relating to probable 100-yr floods and their determination.

The probable 100-yr flood is slightly greater than the flood which will probably be "equalled or exceeded" in a period of 100 yr. Either unit can be used, but the writer prefers to use the former. An actual 100-yr flood for any station is the maximum flood that occurs at that station during some period of 100 yr. More than 90% of actual 100-yr floods will lie between the limits of 80% and 120% of the corresponding probable 100-yr floods.

An excellent study of storm rainfall was made by the Miami Conservancy District[6]. From the U. S. Weather Bureau records the probable 100-yr rain-

NOTE.—The Progress Report of the Committee on Flood Protection Data was presented at the Annual Meeting, New York, N. Y., January 15, 1936, and published in February, 1936, *Proceedings*. Discussion on the report has appeared in *Proceedings*, as follows: April, 1936, by Messrs. Robert Follansbee, and LeRoy K. Sherman.

[5] Colonel, Corps of Engrs., U. S. Army, Detroit, Mich.

[5a] Received by the Secretary, March 28, 1936.

[6] Technical Repts., Pt. V, Miami Conservancy Dist.

fall can be obtained for any locality with a fair degree of accuracy (probable error of less than 10%). The probable 100-yr run-off for a given station can be defined as the run-off which will result at that station if a 100-yr storm occurs above the station, provided the conditions at the beginning of the storm are such that the seepage during the storm will be near the minimum for that locality. It would be possible in certain cases, for an experienced observer to make a tentative selection of actual 100-yr floods, based on the foregoing definition. For example, in the March, 1913, storm in Indiana and Ohio, actual 100-yr floods occurred generally in the area where the total storm precipitation was 9 in., or more, and smaller floods occurred where the precipitation was less than 9 in.

There are several methods for determining the probable 100-yr flood. One of the simplest to apply is the width formula[7]. When used for this purpose by an experienced observer the width formula is fairly accurate, but the results should be checked by any of the other methods which can be applied in a particular case. By means of the width formula the writer has compiled a list of 188 actual 100-yr floods in the United States. Fifty-two of the floods have been checked by precipitation records and practically all of them could be similarly checked by making a search in the U. S. Weather Bureau records. Fifteen of the floods are shown to be the largest that have occurred at the particular stations in a period of 100 yr, or more; and twelve of them have occurred in Pennsylvania, where there appears to have been more research of unofficial records than elsewhere.

Several methods have been proposed for applying the principle of mathematical probabilities to river discharge records in order to determine probable 100-yr floods. Although each of the proposed methods has certain advantages, the results are generally about the same. The writer prefers the Foster method[8]. A Foster determination based on a river record of 35 yr, or more, will have a probable error not greater than 10 per cent. Foster determinations with shorter records (20 to 35 yr) will give values, about one-half of which will be fairly accurate, and the other half will be low, and not satisfactory. The preceding statements are based on more than 100 determinations, 35 of which were for rivers included in the list of actual 100-yr floods. The Foster determinations could be extended at the present time, and the method will increase in value as longer records become available.

The unit-graph method can be used to determine the probable 100-yr flood. This method and its application will undoubtedly be improved as it comes into wider use, and its value will be correspondingly increased.

To summarize, there are three independent methods of determining probable 100-yr flood values:

(1) The width formula, which does not require a river record at an individual station; but the coefficient in the formula must be determined from adequate records of a group of rivers;

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[7] *Engineering News-Record*, June 21, 1934.

[8] "Theoretical Frequency Curves and Their Application to Engineering Problems", by H. Alden Foster, M. Am. Soc. C. E., *Transactions*, Am. Soc. C. E., Vol. LXXXVII (1924), p. 142

(2) The Foster method (or some other similar method), which is based on a record at an individual station; the record must cover at least 35 yr for this method to be satisfactory as a direct check; and,

(3) The unit-graph method which is based on a careful study of an especially selected storm and its run-off.

Each method has certain advantages and certain limitations. Methods (1) and (3) do not require long records, and they can be applied to almost any river station at the present time. Results obtained by checking with two or three of the methods should be quite accurate.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

EQUITABLE ZONING AND ASSESSMENTS FOR CITY PLANNING PROJECTS

PROGRESS REPORT OF COMMITTEE OF CITY PLANNING DIVISION

Discussion

BY GEORGE H. HERROLD, M. AM. SOC. C. E.

GEORGE H. HERROLD,[5] M. AM. SOC. C. E. (by letter).[5a]—Probably the best measure of the equity of a special assessment is the number of assessment delinquencies and the circumstances under which assessments become delinquent. Using these measures it appears quite clearly that present methods of apportioning benefits for local improvements are inequitable. The Committee on Finance and Taxation of the U. S. Chamber of Commerce has compiled data showing that, in 1931, cities with 10% or less of their general tax list delinquent frequently had 20% or more of their special assessments on delinquent lists. This conclusion is also borne out from the partial or preliminary report of the Committee on Tax Delinquency of the National Tax Association.[6]

In St. Paul, Minn., a study of benefit assessments was conducted in 1935 as a Work Relief Project. A carefully prepared questionnaire was sent to officials in all cities with a population greater than 30 000. From the answers, it was possible to analyze the methods used in 153 cities. It will be a long time before the analysis is complete, but there can be but one conclusion, and that is, that the equitable zoning of assessments is impossible by present methods. In fact, such assessments of benefits are so far from actual conditions as to prove conclusively that the basis of the assessment was simply the "enthusiasm of hope."

Correction of such methods may be achieved in several ways: (*a*) Adjust present methods so that they take into account all the known developments

NOTE.—The Progress Report of the Committee of the City Planning Division on Equitable Zoning and Assessments for City Planning Projects was presented at the meeting of the City Planning Division, New York, N. Y., January 15, 1936, and published in February, 1936, *Proceedings*. This discussion is printed in *Proceedings* in order that the views expressed may be brought before all members for further discussion of the report.

[5] Member, State Planning Board; Managing Director and Engr., The City Planning Board, St. Paul, Minn.

[5a] Received by the Secretary April 20, 1936.

[6] *Proceedings*. 25th National Conference, 1932.

of benefit; (*b*) provide legal safeguards against the worst dangers of special assessments; or (*c*) use other means to supplant or assist the special benefit assessments levied.

In adjusting present methods so that they take into account all the known differences of benefit, it must be made possible to use such ideas as "benefit districts", "zones of benefit", and "ratio of benefit." These concepts would constitute a framework or general structure for future assessment methods. The mechanical application of the front-foot rule against abutting property is an undeveloped assessment structure that cannot be expected to differentiate and measure, separately, a complex pattern of benefits. To achieve adequate measurement more is needed than a developed structure of apportionment. There must be guides to assist in finding the structure or framework of the benefit theory to the specific circumstance of the improvement. There must be means of determining the boundaries of special benefit, of distinguishing different types of benefit, and of accounting for the factor of value.

It is true, of course, that rigid restrictions in laws attempt to prevent equitable assessments. For instance, Pennsylvania and Tennessee are still tied to the simple undeveloped front-foot apportionment by State law. On the other hand, some States are trying to encourage more developed methods by granting specific authority to set up benefit districts and zones with different ratios of benefit, such as in California. Furthermore, some States have no restrictions in their statutes and permit the local assessor to develop his own methods subject to review by the Courts.

Studies that have been made in St. Paul would indicate that administrative officials dislike to admit what their system is for determining benefits. This attitude, of course, indicates a lack of clarity of thought on the subject and is quite good proof that the officials are following some precedent—a method that had been used before and possibly had been supported by the Courts.

Possibly one of the reasons for continuing methods that are not equitable in those States where some latitude is given as to methods used, is the fact that the burden of proof is always placed on the one who contests the assessment rather than on the agency spreading the assessment. It is apparent from the bibliography compiled in connection with this study, and from the questionnaires answered, that little thought is ever given to the changing values of all things. When an improvement is financed, the cost of which is to be amortized over a long period of years, the promoters cannot foresee, because of changing economic conditions, whether the improvement is to be a betterment or a detriment from the economic standpoint. The cost of materials varies from year to year and the value of land fluctuates between good times and bad times, and even the value of a beautiful park may change because of changes in social ideas.

In the opening or widening of streets for city planning improvements, it is apparent that the value of the land required for the street was created by the people of the city and not by the owners of the property, and there would seem to be only one clear thought in this problem.

A study of the 153 cities, the data from which were used by the Committee, leads to the conclusion that the human element, in making assessments of

benefits, is so far from being infallible, that the time has come to adopt some more scientific method. Long-time planning, with capital expenditures, are bound to continue and increase in number, and if engineers are not wise enough to determine where the benefit lies in a equitable way, time will help in the solution because the increase in values in one part of the city and the decrease in values in another part, will be the reflection of the value of the general improvements made through the years.

Engineers seeking new fields of endeavor may find it in the determination of benefit assessments, because it is really an engineering problem.

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AMERICAN SOCIETY OF CIVIL ENGINEERS

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DISCUSSIONS

PRINCIPLES TO CONTROL GOVERNMENTAL EXPENDITURES FOR PUBLIC WORKS

FIRST PROGRESS REPORT OF COMMITTEE OF ENGINEERING-ECONOMICS AND FINANCE DIVISION

Discussion

By Messrs. Ivan C. Crawford, and Samuel B. Folk

Ivan C. Crawford,[17] M. Am. Soc. C. E. (by letter).[17a]—The principles, suggested by the Committee, "to guide the Federal Public Works program" are to be applied to both Federal and non-Federal projects alike. A non-Federal project may be defined as one sponsored by a State, or sub-division thereof, the sponsor bearing a portion of the total cost with the Federal Government contributing the remainder as a grant.

Principle (1) proposes the establishment of a "non-partisan Federal agency" to formulate a Federal public works program. Obviously, the first hurdle to be taken is that of securing authorization from Congress to set up such an agency and confer upon it the suggested powers. Experience teaches that in times of extreme emergency Congress freely delegates its powers in matters of this kind. Under normal conditions such delegations of authority are limited and guarded more zealously. Certainly, the experiences of the past few years provide many reasons why an organization of this nature should be created.

In so far as strictly Federal projects are concerned, the "Program Authority" should function smoothly and rapidly because there are relatively few bureaus to deal with, and these agencies are operating continuously. The personnel is experienced. Plans may be outlined far in advance of an emergency.

The relationship between the suggested Authority and non-Federal public works presents some interesting questions which must be taken into consideration. Recent experience has demonstrated the necessity of an effectual de-cen-

Note.—The Progress Report of the Committee of the Engineering-Economics and Finance Division on Principles to Control Governmental Expenditures for Public Works, was presented at the Annual Meeting, New York, N. Y., January 15, 1936, and published in February, 1936, *Proceedings*. Discussion on this report has appeared in *Proceedings*, as follows: April, 1936, by Messrs. Edward W. Bush, Fred Lavis, and Horace H. Sears.

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[17a] Received by the Secretary March 20 1936.

tralization if a non-Federal public works program is to make its influence felt rapidly and efficiently. State or regional sub-divisions must be entrusted with the duty of examining the engineering, financial, and legal data submitted by States and municipalities. Furthermore, these local organizations should have power to pass judgment on projects falling clearly within the formulated rules. Thus, the headquarters at Washington would pass on border-line cases, arrange priorities, and, in a general way, check on projects recommended by the local sub-divisions.

Principle (2), at least that part of it referring to the use of private capital, should be somewhat elastic. In 1933, private capital flowed reluctantly and slowly into municipal projects, even those most worthy and non-controversial. Water and sewerage systems, both new construction and replacement, urgently needed by municipalities with good financial rating, had much trouble in securing funds from private sources at reasonable interest rates. A decision as to whether or not a project is to be financed by private or by Government capital must necessarily take into account the prevailing financial conditions and, therefore, the standard by which judgment is made, will vary from time to time.

The reference to school authorities and school buildings under Principle (2) implies that an agency far removed from the locality applying for a loan and grant is better able to determine the sufficiency of a plant than the applicant itself. Under the existing scheme of government it seems that judgment of the residents of a school district should be decisive as to the needs of the district. The financial state of the district together with information concerning taxable wealth, tax rates, school population, and unemployment should furnish the Federal agency with sufficient data on which to take action. Two years of experience in this work convinces the writer that school districts do not contract debts for new school buildings unless there is a real need for them.

Little exception can be taken to Principle (3) when it is considered as a general statement. In the writer's opinion, the literal application of this principle as a hard-and-fast rule would obstruct progress in national life to a marked degree. If, however, before rejecting a project "which would duplicate, or impair the value of, existing property and facilities that afford adequate service to the public at reasonable rates", there is to be an "impartial and competent investigation" then most of the objection to the rule might be eliminated. As matters now stand, the public is bombarded with propaganda from interested and opposing sides, with no neutral party possessing the confidence of the public to give impartial advice.

For strictly Federal works, the setting up of an order of relative merit should not be impossible of attainment. Non-Federal projects, however, again present a problem when Principle (4) is considered. Is a water supply system in Utah to have precedence over a sewage disposal project in Louisiana? Which will confer the greater benefits on the nation? Title II of the National Industrial Recovery Act of 1933, in Sections 202 and 203, names certain types of public works which are eligible for Government aid. The objective sought by Principle (4), so far as non-Federal works are con-

cerned, could more easily be attained, it appears to the writer, if the different categories of public works were listed in preferred order, somewhat in the manner used in the sections of the National Industrial Recovery Act.

This Committee has approached a controversial subject in a straightforward manner and has produced a very worth-while report. Study of the subject should be continued actively not only because of its closeness to the civil engineer, but also on account of its importance to the public generally.

Samuel B. Folk,[18] Assoc. M. Am. Soc. C. E. (by letter).[18a]—It is hoped that the Committee will amplify the seven broad principles of this interesting report, stating the underlying philosophy, so that every one will know exactly where it stands on the elusive question of public works. This is an intensely live subject, and a thorough discussion and much caution are necessary before there is established, too definitely, any principles so broad that they mean little, or so narrow that the action of the Committee will be regretted in a year or two.

The writer feels that the subject is progressing so rapidly that new ideas are evolved almost before the old ones can be assimilated. Whether the new theories are valid cannot be determined without thorough discussion and (perhaps) some experimentation. The engineer, of course, will not revolt at a scientific experiment, as other men will, because engineering was established on tests as well as theory.

Principle (1).—To base the project on "sound economic consideration exclusively" is a fine beginning, but might be interpreted as a statement too conservative for a body of engineers awake to the necessity of a real public works program sufficient to counterbalance the lack of private construction. It is difficult to get reliable estimates before they are out of date, but, at present, it seems that public works will get not \$1 000 000 000 as originally intended, but only one-third of this sum, the remainder being diverted to other purposes—mostly to the Works Progress Administration (WPA) which, unfortunately, is generally confused with the Public Works Administration (PWA). The total income of the nation has been cut from a peak of approximately \$80 000 000 000 per yr before the depression to about \$40 000 000 000 or \$50 000 000 000 per yr for the last several years. Many industries are now producing on a basis comparable with that of 1928–29. The one contrary example is construction—both private and public. If economic recovery can be attained by Government spending, then public works should be one of the first means of attacking the problem. An allotment of \$5 000 000 000 to \$10 000 000 000 for this purpose is likely to frighten the budget balancers, but, if this will expand the annual income again to approximately \$80 000 000 000, the cost will be cheap. Furthermore, Federal public works funds have been more than matched by local money. This practice should be continued and is a better criterion of the program than the "sound economic considerations." If the project has a useful character

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[18a] Received by the Secretary March 27, 1936.

it will be sufficient. No local community will squander its own money for "white elephants" even if the Federal Government pays one-half the cost. Furthermore, the "sound economic consideration" is incompatible with the second principle, as will be shown subsequently.

To formulate such a program would be quite a task for a body of the character of the United States Supreme Court. This suggestion was probably made to assure that the body would be permanent and non-partisan. However, theories of public works, national ideas of types of projects, and changing conditions affected with the public interest, demand a type of board which is keenly alive to the public weal. A further excellent requirement of the Committee's report (that plans be revised periodically in the light of new developments) leads to the suggestion that the board be constituted like the Interstate Commerce Commission with eleven members whose terms expire at different times. If it is felt that such a board would be too large and unwieldy there is plenty of precedence for a smaller size; for example, the Federal Communications Commission has seven members, and the Federal Power Commission, five. A board the members of which are appointed for definite terms (say, 10 yr) is more likely to examine, critically, any plan submitted and revise periodically any national program which has been adopted. Ten years is not an unreasonable length. Comptroller-General J. R. McCarl was appointed for fifteen years. Furthermore, young and progressive engineers get some consideration on a board the duties of which are formulative rather than negatory. The writer feels, therefore, that the Committee report will receive more favorable comment if the United States Program Authority is organized as a commission, with the United States Senate confirming all appointments.

Principle (2).—The definite public need is a highly commendable requirement. Any project passed by the United States Program Authority, or the present Public Works Administration, would no doubt be a benefit to the economic or social status of society. Whether it will be possible to rate the projects in accordance with their relative merits remains to be seen. How would the Boulder Dam compare with the Tennessee Valley Authority?

I think this is slightly dogmatic.

Although it is true that projects can be divided into two classes, (*A*) those delivering services that produce revenues, or economies; and (*B*) those whose benefits are social or general and not amenable to economics; yet, the "sound economic consideration" of Principle (1) limits deliberations to Class (*A*) since sound economics connotes no social or intangible value. Hence, this requirement is incompatible with Principle (1).

One example of well-established public need is housing. From the F. W. Dodge reports it is seen that residential contracts for three years (1932 to 1934, inclusive) averaged about one-tenth of the 1928 total (\$2 800 000 000). The U. S. Department of Labor reported that between 1930 and 1933 there was an increase in population in 257 cities of about 600 000 families but the total residential construction provided for only 150 000 families. Further losses are caused by fire and demolition—a considerable number. This does not mean there is no need for housing. It merely means that a large class of American people cannot afford decent homes unless the Federal Govern-

ment subsidizes low-rental housing. The Committee would bar this practice, however, under Principle (1), "sound economics", and under Principle (2), Class (*B*), wherein housing and slum clearance are labeled as "local" problems that may affect neighborhood property unfavorably. Before adopting this report the Economics-Engineering and Finance Division should consider carefully whether low-rental housing and slum clearance (and the two are separate and distinct problems which should not be confused) are any more local problems than relief or agriculture, remembering that oranges from Florida compete in Ohio with those from California, and that unskilled labor from Alabama treks to Chicago, Ill., and Detroit, Mich., for jobs. Can the Division afford the criticism of anything but an integrated attack on the intricate problems of a vast country in an automatic-machine age?

The writer feels that civil engineers whose major interest lies in the construction industry must concern themselves with advancing the fundamentals of housing and, if at all possible, should develop new designs and methods of fabricating decent houses. Those who are "scanning the horizon" for an industry or a product to lift the country from the depression should consider the possibility of a much needed reform in housing types, where mass production can contribute to bath-tubs, door-knobs, furnaces, kitchen appliances, and exterior panels. This is mostly a social benefit which does much for research and industry and contributes immensely to fighting disease, fire, and crime. However, some estimate can be made of the economies produced, and some revenues now used for such purposes would be available for other services. Hence, the division between Principle (2) Classes (*A*) and (*B*) is not definite.

A study in 1934 in Cleveland, Ohio, by Howard Whipple Green, for the Public Works Housing Division, showed that one area contained 2.47% of the population, but required fire protection costing 14.4% and police protection costing 6.5% of the taxes. The area requires \$1 962 000 tax money compared to \$785 000 for the average area of equal population. Costs of delinquency, premature death, and penal and welfare institutions for this area must be added to this total. One may conjecture on the percentage of assessed taxes collected from this area compared with the average.

In the last sentence of the "Remarks" concerning Principle (2) the Committee seems to express a feeling that no housing or slum clearance should be undertaken except by local communities because it "may affect neighboring property unfavorably, and they always raise troublesome questions best handled by the localities themselves." This appears to be a good way to prevent anything from being done, since the local communities do not have the funds, have grown up too close to the slum areas to evaluate them, have not the perspective background, or experience, of a Federal board, are inadept and novices at attacking the problem, and probably are controlled by the real estate speculators who will not "play" unless they get "a cut." These are harsh words; but they are reasons why towns cannot do this effectively. As for "affecting neighborhood property", no progress is ever made without affecting present methods. This topic is so obviously the issue that the Committee has made it a part of Principle (3).

Some highways ... railways

Principle (3).—Private industry has no scruples about introducing innovations and new inventions which bankrupt other private industries if there is profit in them. No examples are necessary here, but it might be pointed out that no new highways could be built under the Committee's platform because of the adverse effect on the railroads. It certainly would "impair the value of existing property." All progress is based on making the old property obsolete, undesirable, or inefficient. David Cushman Coyle, M. Am. Soc. C. E., states[19] that a people can have as much progress as they can stand capital loss; yet a major part of the thesis of the Committee seems to be to prevent progress giving a number of examples of what would be prevented by the adoption of this report.

"Legitimate investment", "reasonable basis", and "adequately provided" are terms used in Principle (3) which need an economist to interpret. No service is adequate if a better substitute can be had for less cost. If no public building can be undertaken as long as there is an empty house or a vacant storeroom that might be used (for a postal sub-station, or an art gallery), the country will never arrive at the time when it will need a public works program.

The Committee conjures that if a new and more economical plant replaces an old one, "the loss of value of the old plant" will mean that wealth has been destroyed; but wealth is not defined. Coal and other resources may be conserved—a further gain to humanity. Some one has been injured but the country as a whole is richer—it can produce more goods with less sweat, and all that is necessary is to increase consumption by lowering prices. Any economic justification must be made on the basis of the community, not on the owners of competing services.

Just why the Committee felt the urge to inject the statement that "the scheme of creating 'yardsticks' for public utility plants is unnecessary, expensive, and unjustified" is not clear. Those who have studied the utility field may not agree with this point of view and perhaps conclude that the report is valueless if this is a sample. The fact that the information about municipal plants is public does not seem to solve the difficulties. Furthermore, such projects do not cross county and State lines to make an economic unit of a large area, like the Tennessee Valley Authority. In this respect they are not competing on the same basis as a large utility with its subsidiary corporations.

Principle (4).—Under this heading, the Committee suggests classifying the projects in the order of relative merit based on national rather than local benefits, but ends with an incompatible proviso that projects remote from large centers of population would possess less merit during emergencies. Such a requirement would make any project such as the Tennessee Valley Authority difficult to consider since no large centers of population are affected. Observe that a public works administration is to function only during depressions; that such an area has less merit during emergencies; and that the public works administration will have no appropriation during prosperity. It follows therefore, that "projects remote from large centers of

[19] "Brass Tacks", National Home Library Foundation, Washington, D. C., p. 14.

population" will never get public works. It must be remembered that worthy projects cannot be taken to the unemployed. On the other hand, many believe relief should be divorced from public works. That might be a possible preamble to an amended Committee report rather than a "hedging" under Principle (5).

Principle (6).—Under this heading, the Committee states the foregoing philosophy very well. However, although "planners must recognize that there is a limit to the financial load that the nation can carry", it must not be assumed that Federal borrowing from the banks is to be the order of the day. Few engineers can qualify yet as financial advisers to the Government (although they might do as well as some others). Some more advanced thinkers believe a deficit must be run during depression years and a surplus during booms. At any rate more taxation from those who have, should be expected any time in the future.

Principle (7).—The schedule-of-estimate costs in Principle (7) needs much discussion. Certainly the Government need not carry insurance. Its projects are so diverse and its capital so large that it may well carry the risk itself.[20] The public schools of Ohio receive about $1 back for every $16 of insurance premium.

The Committee suggests that schools have spent an exorbitant sum. As a matter of record the cost of schools has never exceeded 4% of the national income. It was 3.35% in 1930, whereas 4.52% went for life insurance, 2.22% for building construction, and 15.15% for passenger automobiles.[21] This is not an undue proportion for so vital a need as the education of those who must plan even more diligently if the present generation fails.

[20] "State Insurance of School Buildings", *Nation's Schools*, Vol. 16, No. 4, October, 1935, and "Ohio School Plant Insurance", *Educational Law and Administration*, Vol. 4, No. 1, January, 1936, by T. C. Holy.

[21] "Facts on School Costs", *Research Bulletin*, National Education Assoc., Vol. X, November, 1932, pp. 205–226.